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HETEROGENEOUS BELIEFS, LEARNING AND MULTIPLICITY IN DIRECTED TECHNICAL CHANGE

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Heterogeneous Beliefs, Learning and Multiplicity in Directed Technical Change*

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Abstract

The adoption of new technologies occurs at vastly different rates, and is often accompanied by heterogeneous beliefs about their future productivity. Due to strategic complementarities in innovation, transitions to new substitute technologies also manifest as coordination problems with their inherent multiplicity. This paper explores the pivotal role of imperfect information and beliefs in technological change, emphasising the reciprocal relationship between innovation pace and beliefs. A tractable continuous-time two-sector directed technical change model incorporating rational learning about the research productivity of new inventions reveals that belief heterogeneity leads to research misallocation, but it can also significantly influence the transition trajectory and equilibrium selection. A calibration to dirty versus clean technologies shows that with enough optimism about new (clean) technologies, the economy can transition away from otherwise locked-in old (dirty) technologies.

Keywords: Bayesian learning, heterogeneous beliefs, directed technical change, multiple equilibria

JEL Codes: C61, D82, D83, O33, O44, Q55

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1 Introduction

The emergence and expansion of substitute technologies is a major determinant of long-run macro-economic growth. However, not all novel technologies are eventually fully adopted – in the sense of replacing old ones. Some technologies stagnate or are abandoned after a short phase of development. Even for inventions that do grow into maturity, the pace of technological diffusion can vary, with some being expanded faster than others (Adão, Beraja & Pandalai-Nayar, 2024).

Examples of such transitions are abundant. After about three decades of rapid growth, nuclear fission power accounted for about 17% of world electricity production at its peak in mid-1990s (Ritchie & Rosado, 2020). This figure has since decreased to about 9% in 2023 following renewed safety and political pessimism. The German nuclear phase-out in particular, stemming from growing fears following the Chernobyl and Fukushima disasters, saw its nuclear share of electricity decrease from 30% in late-1990s to just under 2% in 2023 (Ritchie & Rosado, 2020; Jahn & Korolczuk, 2012), despite it being one of the safest sources of energy (Ritchie & Rosado, 2020)[1]. Projections remain conservative about a possible future rebound unless new substitute technologies, such as modular reactors or nuclear fusion trigger a new transition.

More contemporary examples are the recent rise in the popularity of artificial intelligence (AI) and quantum computing (QC). Private investments in AI and QC increased from about $10 and $0.1 billion respectively in 2014, to $125 and $2.4 billion in 2021 (Maslej et al., 2024; The Quantum Insider, 2024). However both have seen decreased investments from 2021 to 2023 and, as of the writing of this paper, the outlook remains unclear. Opinions about AI range from the viewing it as a fresh engine for growth akin to a new industrial revolution, to a marginal innovation that is merely influential in highly-specialised domains (Grace et al., 2024). Similarly for QC there is limited consensus about the timeline or possibility of "quantum supremacy" over conventional computers (Meige, Eagar & Arribart, 2023).

A crucial example, and the focus of this paper's calibration, is the diffusion of renewable technologies such as hydro-power, solar and wind. As the implications of climate change become more tangible amid growing evidence of the impact of human-made emissions on the environment, a significant transition towards sustainable production seems increasingly inevitable. At the same time, all major green technologies have seen rapidly increasing productivity in the past few decades (Way, Ives, Mealy & Farmer, 2022), often outperforming projections from the International Energy Agency (IEA) and experts (Metayer, Breyer & Fell, 2015; Meng, Way, Verdolini & Diaz Anadon, 2021). Despite these, the transition to clean technologies continues to exhibit a sporadic trajectory, below the key net-zero targets (Popp, Pless, Haščič & Johnstone, 2020; Acemoglu, Aghion, Barrage & Hémous, 2023).

A recurring feature of technological transitions is the presence of heterogeneous

[1] This of course ignores the (yet unknown) probability of rogue players exploiting nuclear technology for perverse military objectives

and often extreme beliefs about the productivity and growth of new technologies. Motivated by this, I build a theory of technological change in the presence of imperfect information, beliefs and rational (Bayesian) learning. Using a tractable directed technical change (DTC) model with heterogeneous beliefs about the future productivity of new technologies, I show that even in the context of strictly rational learning and convergence of beliefs to the truth, initial optimism and pessimism can have a significant impact on whether a new technology is abandoned or reaches mainstream adoption.

Existing theories of technological change commonly assume that entrepreneurs have ex ante perfect knowledge of the productivity of new technologies, and thus their returns to innovation. But as demonstrated by the examples, once a novel technology has emerged, it is initially not clear how productive it can become as a result of continued research and development. The early adopters of an invention often have limited information, i.e. Knightian uncertainty (Knight, 1921), about its future trajectory, as compared to older technologies for which enough time has passed for a consensus to have been formed about its research productivity. Knightian uncertainty can also occur for more mature technologies in the event of information shocks, such as the nuclear disasters which shifted beliefs about atomic energy. These uncertainties can create a space for beliefs, in particular optimism and pessimism but also disagreement and confidence, to influence the direction and pace of technological change.

It is reasonable to expect that higher optimism about the future productivity of a particular technology increases investments in it, thereby accelerating its adoption. However, three key questions remain. First, the longevity of this effect is unclear: on longer time horizons – those relevant to the energy transition for instance – a reciprocal relationship between investments and beliefs exists, as the pace of innovation can feed back into beliefs. That is, higher investments in a technology lead to faster learning about its true efficiency, thereby diminishing the influence of optimism and pessimism. Secondly, it remains an open question whether optimism and pessimism merely affect the speed of a technology's growth or decline, or if they can also affect the direction of technological change, i.e. which innovation is ultimately adopted. This is especially relevant in the presence of multiple, self-fulfilling equilibria caused by strategic complementarities in innovation (Smulders & Zhou, 2022). Such multiplicity may allow beliefs to act as equilibrium selection mechanisms, in line with the lessons from global games literature (cf. Carlsson & Van Damme, 1993; Morris & Shin, 2001). Finally, heterogeneous beliefs can have important welfare consequences: while biased beliefs may lead to R&D misallocation, optimism might have a positive welfare effect through increased investments in more productive technologies.

To address these questions I propose a continuous-time two-sector (old and new) DTC model in the style of Acemoglu, Aghion, Bursztyn and Hemous (2012) with strategic complementarities as in Smulders and Zhou (2022), which leads to multiplicity and self-fulfilling equilibria. In particular, old and new intermediate good firms employ research labour to improve their technology, and these R&D efforts become strategic complements within each sector if the two final goods are strong enough substitutes – in which case the innovation by each firm creates a positive demand externality

that dominates market competition between firms. The novel feature of this model is the integration of endogenous technical change with imperfect information and learning: investors of the new sector are initially uninformed about their firm-specific returns to R&D investments and instead rely on possibly biased prior beliefs. They then gradually learn about the new technology through Bayesian learning from noisy signals. The speed of learning depends positively on the total research effort in the new sector. As such, this model includes measures of optimism/pessimism, individual confidence as well as aggregate disagreement and tracks the evolution of the entire distribution of beliefs alongside the macroeconomic transition dynamics.

A major complication with models of incomplete information, learning and coordination is the presence of higher order beliefs. Although global games (Carlsson & Van Damme, 1993) are tractable in resolving indeterminacy in static coordination problems with imperfect information (Morris & Shin, 2001), their use remains limited in dynamic situations that involve learning over time (see Angeletos, Hellwig & Pavan, 2007; Giannitsarou & Toxvaerd, 2007; Mathevet & Steiner, 2013). I resolve this by allowing firms to have perfect knowledge of economic aggregates and the distribution of others' beliefs. Because of this, higher-order beliefs do not matter, the model remains tractable and is fully summarised by a system of ordinary differential equations. This does not come at a cost to the internal consistency of firms' information structure[2]: the knowledge of aggregates or others' beliefs does not give additional information about a firm's own productivity.

Using this model I study how heterogeneous beliefs, specifically in the form of optimism or pessimism about research productivity, can impact the direction and pace of technological change. Firstly, I show that belief heterogeneity, irrespective of average optimism or pessimism, causes research misallocation as more optimistic firms overinvest, and redirect research labour away from pessimistic firms with high marginal productivity. Then, using a calibration on the energy transition, I show that if the new (clean) technology is a strong enough substitute for the old (dirty) one, an adequate amount of optimism about it enables the economy to transition away from otherwise locked-in old technologies towards the full adoption of the new ones. Thus even in the absence of aggregate uncertainty, beliefs can act as a coordination device in a model with multiplicity.

When investors have perfect knowledge of R&D productivity, the model predicts that if the new sector starts out much smaller than the old one, strong strategic complementarities in the established technology will lead to a unique outcome: the economy fully adopts the old technology, and the new sector fails to grow. For intermediate initial size of the new sector, strategic complementarities in R&D are comparable in both sectors, thus the model exhibits path-indeterminacy: both the adoption of old and new sectors are possible in a self-fulfilling manner as in (Smulders & Zhou, 2022). Therefore in the absence of optimism and pessimism, the transition to the new tech-

[2]For instance Baley, Figueiredo and Ulbricht (2022) rely on the assumption that agents are not fully rational: they cannot make direct inference (which results in immediate learning) on their noiseless state evolution.

nology is only guaranteed as the unique equilibrium outcome when the new sector is already large compared to the old one. My model provides a mechanism for the take-off of small new technologies, driven by initial optimism: I show that if beliefs about the new technology are optimistic enough, a full transition from an initially small new sector can be possible or even guaranteed as the unique equilibrium outcome. The key mechanism is as follows: The initial amount of optimism gradually decays as investors learn and their beliefs converge to their true productivity of innovation. But the new sector can grow large enough during the transition period, while beliefs are still optimistic, thus ex-post justifying a complete transition.

While optimism can help the economy transition towards the progress of an initially-backward sector, this effect is strongly mitigated by learning. Namely, the more extreme the beliefs of the economy about research productivity are, the faster these beliefs converge towards the true amount of productivity due to learning, thus diminishing the effect of optimism. This fast convergence occurs mainly due to the assumption of Bayesian learning: the further prior beliefs are from the truth, the faster they converge to the truth. The learning speed is also endogenously amplified by investment: the more optimistic firms are, the more they innovate, which leads them to learn faster. The outcome is that the amount of optimism required to enable or guarantee a transition is convex in both the initial size of the new sector and its true productivity. This implies that a relatively moderate amount of optimism is effective in facilitating a transition if the new sector is initially not too small and/or the new technology is relatively more productive. Conversely, the model implies that an extremely high amount of optimism is required for a transition towards an initially small and relatively unproductive new sector, thus limiting the scope of "bad" research bubbles (Komarov, 2020).

To my knowledge, this is the first model that integrates the benchmark DTC framework with internally consistent belief heterogeneity, rational updating, and feedback between economic outcomes and belief dynamics. The model is characterised by its controlled information structure: instead of relying on autonomous or ad hoc learning rules, firms receive only one biased signal initially and then update their beliefs strictly through Bayesian learning. They possess perfect knowledge of economic aggregates, meaning that beliefs only influence their understanding of their own productivity. The persistence of beliefs is also automatically lowered through faster learning caused by higher investments. Even within this rigorous framework, the model demonstrates that beliefs significantly influence the direction and pace of technological change. This implies that relaxing any of these assumptions – such as allowing for higher-order beliefs, behavioural frictions, bounded rationality, or multiple information shocks – would likely further increase the importance of beliefs in determining economic outcomes.

Related Literature This paper is firstly related to the DTC models in the tradition of Acemoglu et al. (2012), Acemoglu, Akcigit, Hanley and Kerr (2016) and Hassler, Krusell and Olovsson (2021) with strategic complementarities and multiplicity (Van der

Meijden & Smulders, 2017; Smulders & Zhou, 2022). Van der Meijden and Smulders (2017) study the complementarity between resource conservation decisions by resource owners and investments in resource-augmenting technologies by innovators, which leads to a coordination problem with sunspot equilibria. Smulders and Zhou (2022), to which this paper is most closely related, relax the common one-period patent assumption in the literature, thus stressing the multiplicity and path-dependence in the transition dynamics of technological change. I contribute to this literature by integrating imperfect information and (rational) learning in a general equilibrium directed technical change model. By explicitly modelling the spillovers between beliefs and the macroeconomy, I show that beliefs can act as an equilirbium selection mechanism and trigger the emergence of small sectors.

This paper is also related to the literature on the effect of beliefs on investment behaviour, particularly in the context of the green transition. On the empirical side, using surveys of U.S. retail investors, Ceccarelli and Ramelli (2024) find a causal effect of climate transition optimism on increased investments in green projects. Giglio et al. (2023) confirm the effect of investors' beliefs about sustainable equity returns on their portfolio share of sustainable funds. Ramadorai and Zeni (2024) find that firms' reported beliefs about climate regulation strongly influence their planned and actual abatement activities. Of note is also the aforementioned work of Metayer et al. (2015) and Meng et al. (2021) who find that both the IEA and technology experts have underestimated the green technology progress, while Giglio et al. (2023) find some investors to expect sustainable equities to underperform the market. My paper adds to this literature by providing theoretical justification in a general equilibrium model that pessimistic beliefs about the returns to green investments can be considered as an important contributor to the slow green transition, thus advocating for higher optimism about these technologies. Thus this paper is complementary to theories stressing the importance of trust in climate policy and calling for higher policy commitment (Campiglio, Lamperti & Terranova, 2024).

2 Model

The model in this section is based on Smulders and Zhou's (2022) model of directed technological change in two sectors, new and old, where monopolistic intermediate good firms hire researchers in order to increase the quality of their goods and increase profits. The key new aspect of this model is that these firms have imperfect information about the productivity of R&D expenditures, and instead use their (noisy) quality process as an informative signal in order to update their beliefs through Bayesian learning. In the description of the model I suppress the time subscript where it does not cause confusion.

2.1 Setup

Final Good Producers Production of final good in the economy takes place at two competitive sectors (new n and old o). The production technology for sectors $j \in \{n, o\}$ is Cobb-Douglas using production labour L_j and a unit mass of sector-specific intermediate goods x_{ji}, $i \in [0, 1]$, with quality q_{ji}

$$Y_j = L_j^{1-\alpha} \int_0^1 q_{ji} x_{ji}^{\alpha} \mathrm{d}i \tag{1}$$

Final good producers maximise profits given by

$$\pi_j = P_j Y_j - w L_j - \int_0^1 P_{ji} x_{ji} \mathrm{d}i \tag{2}$$

where P_j is the price of final good of sector $j \in \{n, o\}$, w is the nominal wage, and P_{ji} is the price of intermediate good i in sector j.

Households Households derive utility from a constant elasticity of substitution (CES) composite final good with elasticity of substitution σ between the two sectoral goods. They supply one unit of labour inelastically at wage w, and invest in assets $\mathcal{A}$ with return r:

$$\begin{aligned} C &= \left[C_n^{\frac{\sigma-1}{\sigma}} + C_o^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \\ U &= \ln(C) \\ \max_{\{C_n(t), C_o(t)\}_{t=0}^{\infty}} \mathcal{W}_0 &= \int_0^{\infty} \ln(C(t)) e^{-\rho t} \mathrm{d}t \\ &\text{s.t. } \dot{\mathcal{A}} = r\mathcal{A} + w - P_n C_n - P_o C_o \end{aligned} \tag{3}$$

Intermediate Good Producers There is a mass 1 of intermediate good producers in each sector, such that averages and aggregates are the same in each sector. The quality/technology of firm i in sector j follows the stochastic process

$$\mathrm{d}q_{ji} = s_{ji}^{\beta} Q_j \left[\mu_{ji} \mathrm{d}t + \eta(S_j) \mathrm{d}W_{jit} \right] \tag{4}$$

where $\left\{ W_{jit} \right\}_{t \geq 0}$ are standard Wiener processes independent across sectors j and firms i

$$\mathrm{d}W_{jit} = \varepsilon_{jit} \sqrt{\mathrm{d}t}, \quad \varepsilon_{jit} \sim \mathcal{N}(0, 1)$$

and where s_{ji} is the amount of research labour hired by firm i in sector j; and Q_j, S_j are aggregate technology and scientific labour in sector j respectively

$$Q_j := \int_0^1 q_{ji} \mathrm{d}i \quad \text{and} \quad S_j := \int_0^1 s_{ji} \mathrm{d}i. \tag{5}$$

Furthermore η is a smooth, bounded and decreasing function of S_j with $\eta \to \infty$ as $S_j \to 0$, and μ_{ji} is a constant idiosyncratic parameter denoting the productivity of

firms' R&D activities. This productivity is normally distributed in each sector with mean $\overline{\mu}_j$ and variance b_j^2.

The quality growth process (4) implies two sources of spillovers among the intermediate good producers of each sector. First, the expected growth of any firm's quality depends positively on aggregate technology Q_j. This indicates standard knowledge spillovers: innovation builds on the existing stock of knowledge. The diffusion term also depends positively on the individual producer's research effort s_{ji} and aggregate quality Q_j, which represents the proportional effect of firm and sector size on the volatility of technological growth.

The second important source of spillovers is due to the presence of $\eta(S_j)$ function in the volatility term, indicating that an increase in the sector's research effort has a direct negative impact on the volatility of research output. This can be considered as a form of collaborative stability effect, where increased research labour investment leads to a more stable research environment through shared expertise and collective efforts in the sector. I call these *learning spillovers* since, as will be explained below and in Proposition 1, this leads to a positive effect of higher sectoral innovation on the speed of learning of firms. Similarly if a sector has zero investments in technology, then the quality growth process will be infinitely noisy which completely halts the ability of firms to learn. Moreover, as explained below in Proposition 2, due to inherent linearities in the model, the volatility in the quality growth process does not affect firms' investment decision s_{ji}, implying that the learning spillovers are the only outcome of research volatility.

Additionally $\beta \in (0,1)$ indicates a stepping-on-toes effect in firm's R&D activity that causes diminishing marginal returns to research effort (Porter & Stern, 2000). Specifically, let $\dot{q}_{ji}^e := \mathbb{E}[\mathrm{d}q_{ji}]/\mathrm{d}t$ denote the unconditional expected number of new ideas per unit of time, then from (4) the firm-specific (expected) research productivity can be written as (Bloom, Jones, Van Reenen & Webb, 2020):

$$\text{research productivity} = \frac{\text{expected rate of new ideas}}{\text{number of researchers}} = \frac{\dot{q}_{ji}^e/Q_j}{s_{ji}} = \mu_{ji}/s_{ji}^{1-\beta} \tag{6}$$

which declines as the firm hires more research labour.

Intermediate firms choose amount of production x_{ji} and research labour s_{ji} to maximise net present value of profits:

$$\pi_{ji} = \big(P_{ji} - P_j q_{ji}\big) x_{ji} \tag{7}$$

where the unit cost of production increases with quality: one unit of intermediate good requires q_{ji} units of finals goods Y_j (Smulders & Zhou, 2022).

Incomplete Information and Learning The key property of the model is that each intermediate producer i in sector j has imperfect information about its research productivity μ_{ji} and thus forms beliefs about this parameter by regarding its quality process (4) as a noisy signal. In other words, for each firm the process in (4) functions

both as a real technological growth process, as well as a signal to be used for statistical inference about μ_{ji} using Bayesian learning.

The information structure of each firm is as follows: as well as not knowing the true value of μ_{ji}, it also does not observe the sequence of shocks W_{jit} $\forall t$, but perfectly observes its own quality process q_{jit} and its history. Therefore the firm's relevant private information at time t is summarised by the σ-algebra generated by the firm's own quality process up to time t,

$$\mathcal{I}_{jit} := \sigma\big(\{q_{jiu} : u \leq t\}\big) \tag{8}$$

and we denote by $\mathbb{E}_{jit}$ the expectation operator conditional on this information set, i.e. $\mathbb{E}_{jit}[\cdot] := \mathbb{E}\big[\cdot|\mathcal{I}_{jit}\big]$.

We allow the firm to observe all other parameters and aggregates of macroeconomic variables in both sectors, such as Q_j, S_j that appear in (4), as well as the true distribution of μ_{ji} and beliefs in the cross-section of firms. This assumption of common knowledge of distributions is made in order to abstract from higher order beliefs and focus solely on beliefs about research productivity and learning. In many models, perfect knowledge of aggregates creates an opportunity for the agents to perfectly learn the unknown parameter within an infinitesimal amount of time. Thus a bounded rationality assumption, in particular ignorance of available information, is needed to maintain the role of beliefs and learning in the model. This is not the case here: while succeeding at removing the role of higher order beliefs, this information structure is nonetheless fully consistent with the firms being rational, albeit uninformed, agents. That is, even though firms have perfect information about aggregates, and consequently also perfect foresight about them, this does not give them any additional information about their own research productivity. In other words, even though each firm knows the distribution of research productivity in the cross-section of the sector, it does not know where on this distribution it belongs. Moreover, even though firms may observe that the sector's beliefs about μ_{ji} are distorted away from its true distribution (for instance beliefs being too optimistic), this also does not give them additional information regarding which particular firms, including themselves, are mistaken in their beliefs: "*I know someone is mistaken, but I do not know who*". As such the signal (4) remains their only source of information.

As it is known in linear filtering theory, assuming that each firm's prior belief density about μ_{ji} is normally distributed at time t, the posterior density is also normal at all times, whose evolution is given by the Kalman-Bucy filter equations[3]. Denoting by $\mu^e_{jit} := \mathbb{E}_{jit}[\mu_{ji}]$ the posterior mean, its evolution is given by the stochastic differential equation

$$\mathrm{d}\mu^e_{ji} = \frac{v_{jit}}{s^\beta_{ji} Q_j \eta (S_j)^2}\Big[\mathrm{d}q_{ji} - s^\beta_{ji} Q_j \mu^e_{ji} \mathrm{d}t\Big] \tag{9}$$

[3]see Oksendal (2013, Ch.6) for an introduction to stochastic filtering theory or Kallianpur (1980) and Liptser and Shiryaev (1977) for a more in-depth analysis.

where $v_{jit} := \mathbb{E}[(\mu_{ji} - \mu^e_{jit})^2]$ is the variance of the posterior belief about μ_{ji}, and evolves according to the deterministic Riccati equation

$$\dot{v}_{ji} = -\left(\frac{v_{ji}}{\eta(S_j)}\right)^2. \tag{10}$$

which directly implies v_{ji} converges to 0 as long as $\eta(S_j)$ is finite. That is, as long as there is any investment in sector-j technology, firms' estimate of research productivity becomes increasingly more precise over time due to Bayesian learning. However the speed of this convergence depends on the amount of research effort, indicating learning through experimentation. If a sector stops investing in its technology (i.e. $S_j = 0$), then, since the signal noise η approaches infinity, learning stops completely.

Expanding (9) using (4) gives a time-varying Ornstein-Uhlenbeck process (akin to AR(1) in discrete-time)

$$\mathrm{d}\mu^e_{ji} = \frac{v_{ji}}{\eta(S_j)^2}(\mu_{ji} - \mu^e_{ji})\mathrm{d}t + \frac{v_{ji}}{\eta(S_j)}\mathrm{d}W_{ji} \tag{11}$$

which already points to the normality of mean beliefs in the cross section of firms, as will be formalised in Proposition 1. A useful feature of (11) is that it does not depend on q_{ji}, implying that in this model we do not need to keep track of the distribution of states q_{ji} across firms. This is due to the fact that quality increments $\mathrm{d}q_{ji}$ themselves do not depend on the level of quality q_{ji}, meaning that a firm's current quality in levels, q_{ji}, is not informative about its research productivity μ_{ji}. The drift term in (11) shows that the speed of convergence of beliefs from their current level μ^e_{ji} to the true productivity μ_{ji} is decreasing in the current level of confidence and increasing in technology investments. The higher a firm's confidence about its beliefs or lower the investments in the j-sector technology, the slower beliefs are updated. The diffusion term in (11) implies a similar relationship: the higher the firm's confidence or lower the sector's innovation effort, the less the responsiveness of posterior beliefs to quality shocks.

We make the following assumption regarding the cross-section distribution of firms' priors:

Assumption 1. The distribution of priors in the cross section of firms in each sector satisfies:

a. All firms have the same variance of prior belief about μ_{ji}: $v_{ji0} = v_{j0}\ \forall i$. That is, all firms are initially equally confident about their beliefs about μ_{ji}.

b. The initial distribution of mean beliefs μ^e_{ji0} within each sector is normal, with mean M_{j0} and variance Σ_{j0} and is independent of the true distribution of μ_{ji}.

Assumption 1.a is a simplifying assumption, which together with equation (10) implies that the variance of posterior belief is common within each sector at all times: $v_{ji} = v_j\ \forall i, t$. This, however, implies that while this model includes a measure of absolute over- or under-confidence, it does not include a notion of relative overconfidence

between agents (Broer & Kohlhas, 2022). Assumption 1.b is sufficient to ensure normality of beliefs for all times $t > 0$. Here the assumption of independence of initial beliefs from true value of μ_{ji} is made for simplicity and the model can still be solved if they are assumed to be correlated. This is also a relatively inconsequential assumption because even though at time 0 firm i's belief is independent of its particular research productivity, this is no longer the case for all $t > 0$ following from (11). As expected due to learning, beliefs become immediately correlated with the true value of μ.

Following Assumption 1, Proposition 1 shows that the entire distribution of beliefs, including its dependence on the true distribution of μ's, is Gaussian and characterised by a system of ordinary differential equations. The proof is provided in Appendix A.1

Proposition 1. *Given Assumption 1, sector j's cross-sectional distribution of mean beliefs, μ^e_{ji}, is normal for all t, with mean $M_{jt} = \tilde{M}_{jt} + \overline{\mu}_j B_{jt}$ and variance $\Sigma_{jt} = \tilde{\Sigma}_{jt} + b_j^2 B_{jt}$, which follow the deterministic differential equations*

$$\dot{\tilde{M}}_j = -\frac{v_j}{\eta(S_j)^2}\tilde{M}_j, \quad \tilde{M}_j(0) = M_{j0} \tag{12}$$

$$\dot{\tilde{\Sigma}}_j = \frac{v_j}{\eta(S_j)^2}(v_j - 2\tilde{\Sigma}_j), \quad \tilde{\Sigma}_j(0) = \Sigma_{j0} \tag{13}$$

$$\dot{B}_j = \frac{v_j}{\eta(S_j)^2}(1 - B_j), \quad B_j(0) = 0 \tag{14}$$

Proposition 1 presents a decomposition of the distribution of mean beliefs at each moment in time into two separate and statistically independent components: the correlation of this distribution with initial beliefs – governed by (12) and (13) – and its correlation with the true distribution, governed by (14).

Similar to equation (10), these dynamics imply that as long as $S_j, v_j \neq 0$, $\tilde{M}_j$ and $\tilde{\Sigma}_j$ converge to 0 and B_j converges to 1: as long as sector j hires a positive amount of research labour, over time all firms in the sector learn about their productivities, eventually each firm's belief density converging to a point mass at its true productivity μ_{ji}. The speed of convergence is given by v_j and η: the lower the amount of investments in research, or the higher the confidence of firms about their current beliefs (meaning lower v_j) the slower the learning process becomes, confirming the previous intuition. In summary, M_j, v_j and Σ_j can be regarded as measures of optimism/pessimism, (in)confidence/precision and (dis)agreement in sector j respectively, and B_j is a measure of overall correctness of beliefs. Figure 1 shows an example of beliefs following Kalman-Bucy dynamics where an initially dispersed and optimistic distribution of beliefs converges to the true distribution of μ_{ji}.

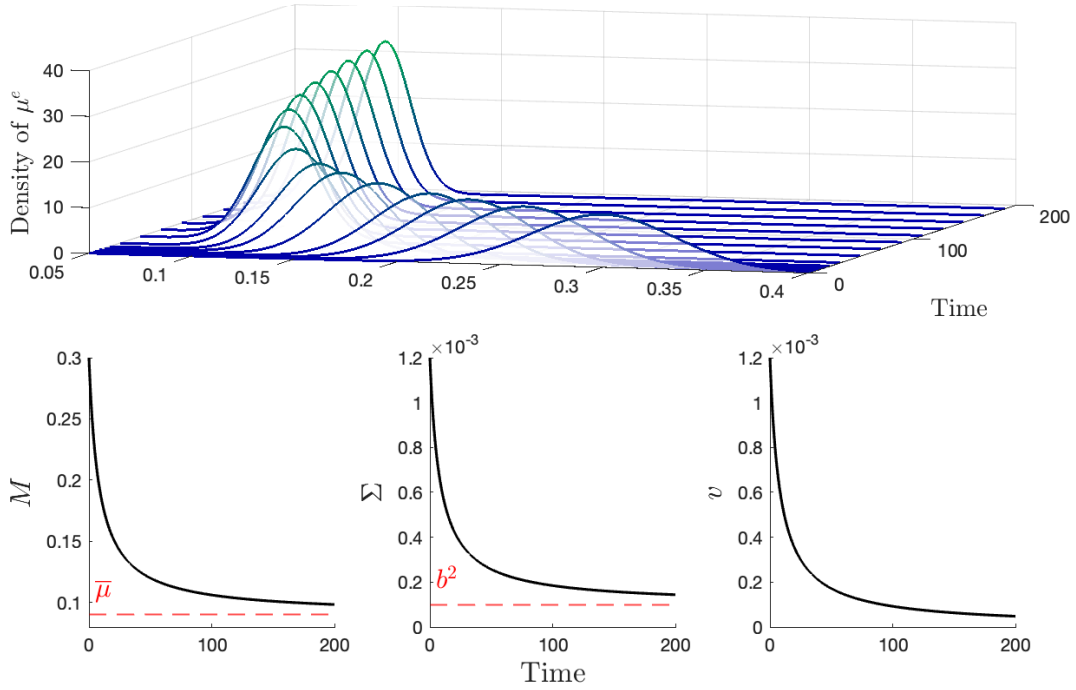


Figure 1: *Top panel*: evolution of distribution of beliefs in the cross-section. *Bottom panels*: evolution of the mean, variance and precision of beliefs. *Parameters*: $\eta = 0.1$, the true distribution of μ has mean $\overline{\mu} = 0.09$ and standard deviation $b = 0.01$, and the initial distribution of beliefs has precision $v_0 = 0.0012$, dispersion $\Sigma_0 = 0.0012$ and mean $M_0 = 0.3$.

Another implication of the Kalman-Bucy belief dynamics is that if there is much more agreement than confidence in beliefs (specifically if $v_j > 2\tilde{\Sigma}_j$) then disagreement momentarily increases before decreasing again[4]. This is because, as seen from (11), lower confidence not only increases the speed of transition towards the correct value of μ_{ji}, but it also increases the volatility of each individual producer's belief process, contributing to higher dispersion in mean beliefs. For low enough confidence, the increased volatility in beliefs dominates the increased speed of convergence towards the true productivities and thus increases disagreement in the short term.

So, for each sector, 4 differential equations (10), (12), (13) and (14) characterise the state of beliefs in the economy. However, these beliefs are not fully independent of the rest of the economy: they depend on macroeconomic variables through the aforementioned learning spillovers. As investment in clean/dirty technologies increases, firms receive less noisy data from their R&D activities, and thus learn faster about the productivity of these technologies.

[4]More specifically, $1/v_j$ denotes confidence, and $1/\tilde{\Sigma}_j$ denotes the degree of agreement.

2.2 Subjective Value of the Firm and Misallocation

In this section we solve the intermediate good producers' intertemporal choice of research labour and use that to characterise the aggregate growth rate of the economy in the presence of imperfect information. Firstly, final good producers' factor demands are given by the standard expressions

$$w = P_j(1-\alpha)\frac{Y_j}{L_j} \tag{15}$$

$$P_{ji} = \alpha P_j L_j^{1-\alpha} q_{ji} x_{ji}^{\alpha-1} \tag{16}$$

The subjective value of an intermediate firm at time t is a function of its economic state q_{jit}, and beliefs dictated by its information set $\mathcal{I}_{jit}$. However, since we know the firm's beliefs are normal, summarised by mean and (common) variance:

$$\mu|\mathcal{I}_{jit} \sim \mathcal{N}(\mu^e_{jit}, v_{jt})$$

then for each information set $\mathcal{I}_{jit}$, there exist pair μ^e_{jit}, v_{jt} such that the subjective value function of the producer can be written as

$$V(q_{ji}, t; \mathcal{I}_{ji}) = V(q_{ji}, t; \mu^e_{ji}, v_j) \equiv V_j(q_i, t; \mu^e_i, v)$$

and defined by

$$V_j(q_i, t; \mu^e_i, v) = \max_{\{s_{iu}, x_{iu}\}_{u=t}^{\infty}} \mathbb{E}\left\{ \int_t^{\infty} e^{-r(u-t)} \left[\pi_j(q_{iu}, s_{iu}, x_{iu}) - w_u s_{iu}\right] \mathrm{d}u \,\middle|\, \mathcal{I}_{jit}, q_{it} = q_i \right\} \tag{17}$$

subject to the law of motion of q_i in (4) with the instantaneous profit function given by (7) and taking factor demands (15),(16) into account. Note that in reality μ^e_{ji} and v_j follow the dynamics in (11) and (10) but the intermediate producer takes them as given when maximising (17), and they are subjectively regarded as constants since the firm cannot be forward-looking about its own learning. Here, the optimal choice of quantity produced is static and is given by

$$x_{ji} = \alpha^{\frac{2}{1-\alpha}} L_j \tag{18}$$

which implies that all firms in the sector sell the same quantity but more productive firms sell at a higher price:

$$P_{ji} = \frac{1}{\alpha} P_j q_{ji} \tag{19}$$

Furthermore, equations (18) and (1) imply that aggregate productivity in each sector is proportional to the aggregate knowledge stock and production labour in that sector

$$Y_j = \alpha^{\frac{2\alpha}{1-\alpha}} L_j Q_j \tag{20}$$

Consequently each intermediate producer's profits are proportional to its share of the technology stock

$$\pi_{ji} = \alpha(1-\alpha) P_j Y_j \frac{q_{ji}}{Q_j} \tag{21}$$

Importantly (21) is linear in the firm's individual quality. This linearity, together with the fact that both $\mathrm{d}q_{ji}$ and $\mathrm{d}\mu^e_{ji}$ in (4) and (11) do not depend on q_{ji} implies that the value function itself is linear in q and, crucially, the shadow value of quality improvements is symmetric across all firms. This is summarised in the proposition below, the proof of which is outlined in Appendix A.2.

Proposition 2. *The value function defined in* (17) *is linear in quality* q_{ji} *and satisfies the Bellman equation*

$$rV_j = \max_s \left\{ \alpha(1-\alpha)\frac{P_j Y_j}{Q_j} q_i - sw + \frac{\partial V_j}{\partial q_i} s^\beta Q_j \mu^e_i + \frac{\partial V_j}{\partial t} \right\} \tag{22}$$

and the shadow value of quality improvements $\lambda_{jit} := \partial V_j(q_i, t)/\partial q_i$ *is common across all firms in sector* j*:* $\lambda_{jit} = \lambda_{jt}\ \forall i$*. Furthermore,* λ_j *obeys the differential equation*

$$\dot{\lambda}_j = r\lambda_j - \frac{\partial \pi_{ji}}{\partial q_{ji}} = r\lambda_j - \alpha(1-\alpha)P_j Y_j/Q_j \tag{23}$$

From the Bellman equation (22), each firm's choice of research labour is given by the first order condition

$$w = \beta s_{ji}^{\beta-1} Q_j \mu^e_{ji} \lambda_j \tag{24}$$

with $\lambda_{jt} := \partial V_j/\partial q$ denoting the aforementioned shadow value of innovation. The optimality condition (24) equates the marginal cost and benefits of research effort. The left hand side is the marginal cost of an additional unit of research labour, $\beta s_{ji}^{\beta-1}\mu^e_{ji}Q_j$ is the *believed* contribution of a marginal unit of research to quality improvements, and λ_{ji} is the present value of each extra unit of quality q. Therefore the right hand side is the believed marginal contribution of research effort to firm's net present value of future profits.

One implication of Proposition 2 and (24) is that the idiosyncratic shocks in firms' quality process (4) only indirectly affect their research labour decisions through the beliefs and learning channel. That is, due to linearity of the value function, firms only take their expected future quality into account, which is symmetric across the firms in each sector conditional on beliefs and does not depend on the randomness in firms' quality process. However, apropos of (4) as an information signal, the stochastic noise does affect beliefs over time, and thus dynamically influences research effort as seen in (24). Proposition 2 and (24) also imply that only the mean μ^e_{ji} of each firm's belief density is relevant for its R&D decision, which is fully summarised by the dynamics in Proposition 1.

The linearity of the value function also sets the model apart from the experimentation literature, further justifying the quality growth process (4). In that literature, due to non-linearities and irreversible investments, firms have an incentive to make some initial investments in order to increase their confidence of beliefs (decrease v_j) even if under their current beliefs investment has negative NPV. In other words, investment becomes a *learning real option*. In this model, equation (11) showed that the

terms that equally affect both the drift and diffusion of the signal, namely individual investments s_{ji} and aggregate technology Q_j, drop out and do not affect the speed of learning. An alternative specification would be to have the noise η as a function of each firm's own investments s_{ji}. However, due to the linearity result above and the fact that research labour is a reversible investment, this extension would not directly affect firms' investment choice. The only outcome of this would be differential rates of learning between firms, not just sectors. This would likely not drastically change the qualitative behaviour of the model, while imposing the complication of non-normal distributions. Thus this is left for future extensions.

From (24), integrating $ws_{ji} = \beta s_{ji}^{\beta} Q_j \mu_{ji}^e \lambda_j$ across sector j we get that firm i's share of total research effort in sector j is given by

$$\left(\frac{s_{ji}}{S_j}\right)^{1-\beta} = \frac{\mu_{ji}^e}{\left(\int_0^1 (\mu_{ji}^e)^{\frac{1}{1-\beta}} \mathrm{d}i\right)^{1-\beta}} \equiv \frac{\mu_{ji}^e}{\mathrm{CES}_{\frac{1}{1-\beta}}(\mu_j^e)} \tag{25}$$

which only depends on its level of optimism relative to a CES composite of beliefs in the cross-section of firms. This implies that, as expected, more optimistic firms also invest more. But the relationship between optimism and R&D effort is nonlinear and when S_j is determined in equilibrium, equation (25) also characterises the density of research effort s_{ji}. Figure 2 shows two examples of the density of s_{ji} and how it relates to the density of beliefs μ_{ji}^e. In particular, in line with observations (Ugur, Trushin & Solomon, 2016), the distribution of investments in R&D can become right-skewed.

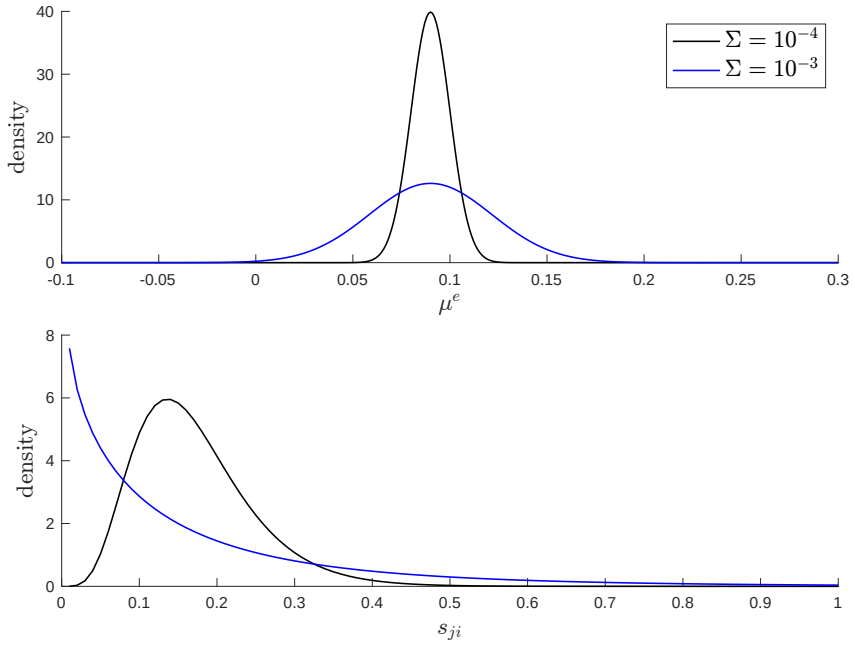


Figure 2: Densities of beliefs μ_{ji}^e and research effort s_{ji}, when $M_j = 0.09, S_j = 0.1714, \beta = 3/4, \Sigma_j \in \{10^{-4}, 10^{-3}\}$

The first order condition for R&D investments (24) implies that the only source of heterogeneity in research effort is due to the heterogeneity in beliefs and because of

diminishing returns, $\beta < 1$, research effort is convex in beliefs. The intuition behind this convexity is that an increase in R&D productivity μ does not affect the marginal cost of research effort, which is the wage rate, but linearly increases its marginal benefit. Therefore the firm finds it optimal to hire more research labour. But due to its diminishing marginal returns, the firm needs to employ an increasing quantity of scientists in order to equate the marginal cost and revenues of its technology investments. Precisely this convexity, and the fact that beliefs are the only source of heterogeneity, imply that when the dispersion of beliefs is increased (keeping the average beliefs constant) the additional investments from high-μ firms dominates the decreased investment of low-μ firms such that the density of technology investments becomes more right-skewed as figure 2 shows. However note that while total research labour S_j is kept constant, sectoral research *output* $\int s_{ji}^{\beta} \mathrm{d}i$ decreases as a result of the increase in the dispersion of beliefs, again as a result of diminishing returns.

Growth and Misallocation Using (4), (5) and (25), aggregate technology growth in each sector j is given by

$$\widehat{Q}_j \equiv \dot{Q}_j / Q_j = \int_0^1 s_{ji}^{\beta} \mu_{ji} \mathrm{d}i = S_j^{\beta} \frac{\mathbb{E}\left[(\mu_{ji}^e)^{\frac{\beta}{1-\beta}} \mu_{ji} \right]}{\mathbb{E}\left[(\mu_{ji}^e)^{\frac{1}{1-\beta}} \right]^{\beta}} \tag{26}$$

In equation (26) and from now on, the unconditional expectation operator denotes averages in the cross section of firms, $\mathbb{E}[x_{ji}] \equiv \int x_{ji} \mathrm{d}i$.

If all firms have the same mean belief about their research productivity, i.e. $\mu_{ji}^e = \mu_j^e \ \forall i, j$, then the aggregate knowledge growth reduces to $\widehat{Q}_j^* = S_j^{\beta} \mathbb{E}[\mu_{ji}]$. On the other hand, if all firms have correct beliefs about their research productivity, i.e. $\mu_{ji}^e = \mu_{ji} \ \forall j, i$, then aggregate quality growth reduces to[5]

$$\widehat{Q}_j^{**} = S_j^{\beta} \mathbb{E}\left[\mu_{ji}^{\frac{1}{1-\beta}} \right]^{1-\beta} \tag{27}$$

Then the relation between these rates of technology growth is given in lemma 1, with proof given in Appendix A.3.

Lemma 1. *Given the same amount of investment in R&D, S_j, aggregate TFP growth in the presence of beliefs is at most equal to the case with homogeneous beliefs, which is at most equal to the case with perfect information: $\widehat{Q}_j \leq \widehat{Q}_j^* \leq \widehat{Q}_j^{**}$.*

This denotes misallocation in research due to heterogeneous beliefs: conditional on total research effort S_j, inaccurate beliefs decrease average productivity growth as

[5]Note that in equations (26), (27) and in the rest of the text, unconditional expectations denote averages across firms, using the law of large numbers.

more optimistic firms over-invest in R&D. This productivity loss may counteract potential welfare gains from more extreme beliefs about a certain technology, for instance higher optimism about sustainable technologies.

It must be noted that equation (26) does not explicitly show the relationship between beliefs and the growth rate $\widehat{Q}_j$, since beliefs also affect S_j in that equation in labour market equilibrium. For the same reason, the fact that beliefs do not appear explicitly in the equation for $\widehat{Q}_j^*$ should not be taken as a sign that the level of optimism does not affect this growth rate. However, since total research effort S_j appears symmetrically in these growth rates, and motivated by lemma 1, we can quantify the degree of misallocation as a result of heterogeneity of beliefs as the ratio $\widehat{Q}_j^* / \widehat{Q}_j$. Figure 3 shows this ratio when all firms have the same true productivity, i.e. $b = 0$, as a function of the dispersion Σ and degree of optimism of beliefs M. As expected, higher dispersion in beliefs results in higher misallocation. However higher optimism results in a lower misallocation ratio as defined here.

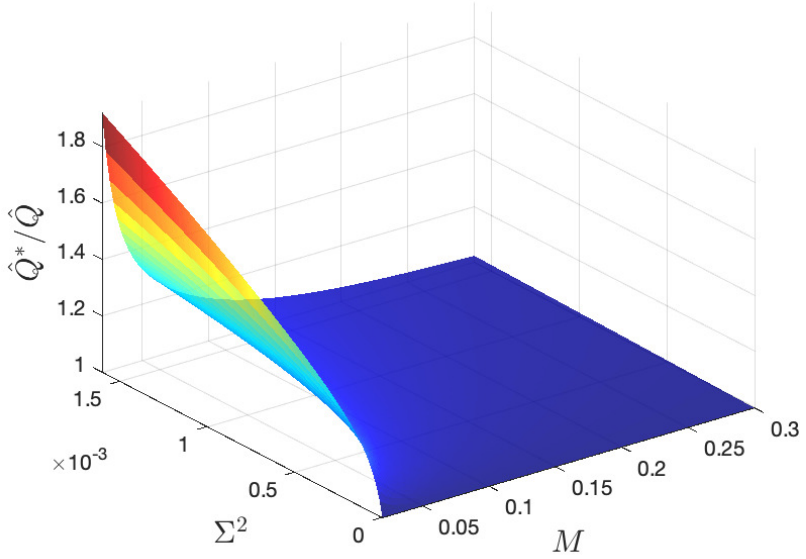


Figure 3: Misallocation due to heterogeneity as a function of dispersion and optimism. $\beta = 3/4$

2.3 Households, Market Clearing and Static Allocation

Households' static demand and Euler equations are given by

$$\frac{C_n}{C_o} = \left(\frac{P_n}{P_o}\right)^{-\sigma} \tag{28}$$

$$r = \rho + \widehat{C} + \widehat{P} \tag{29}$$

where P is the consumption price index given by

$$PC := C_n P_n + C_o P_o$$

And goods market clearing condition, together with (18), (20) implies

$$C_j = Y_j - \int_0^1 q_{ji} x_{ji} \mathrm{d}i = (1-\alpha^2) Y_j \tag{30}$$

And finally the labour market clearing condition is

$$1 = L + S_n + S_o$$

where $L := L_n + L_o$ is total production labour. As in Smulders and Zhou (2022), we define $\theta_j \in [0,1]$ as the relative size of a given technology stock Q_j:

$$\theta_j := \frac{Q_j^{\sigma-1}}{Q_n^{\sigma-1} + Q_o^{\sigma-1}} \tag{31}$$

By this definition, $\theta_o = 1 - \theta_n$, and θ_n, the market share of the new sector, follows the law of motion

$$\dot{\theta}_n = \theta_n (1-\theta_n)(\sigma-1)(\widehat{Q}_n - \widehat{Q}_o) \tag{32}$$

Then the static allocation of consumption and production labour between sectors is fully described by the history-dependent variable θ_c:

$$\theta_j = \frac{L_j}{L} = \frac{P_j C_j}{PC} = \frac{P_j Y_j}{PY} \tag{33}$$

2.4 Research Incentives and Dynamic Allocation

We define m_j as the marginal value of increasing size of sector j as share of total market value

$$m_j := \frac{Q_j \lambda_j}{\sum_k Q_k \lambda_k} \tag{34}$$

and m_n (or equivalently $m_o = 1 - m_n$) can be shown to determine the allocation of research labour between sectors (see Appendix A.4 for derivation)

$$\left(\frac{S_o}{S_n}\right)^{1-\beta} = \frac{1-m_n}{m_n} \frac{\mathrm{CES}_{\frac{1}{1-\beta}}(\mu_o^e)}{\mathrm{CES}_{\frac{1}{1-\beta}}(\mu_n^e)} \tag{35}$$

As such m_n denotes the market value share of the new sector and, given the distribution of beliefs, determines the allocation of *research* labour between new and old sectors. However the allocation of labour between production and research still needs to be determined. For this we define variable z, which is the same for both sectors, as

$$z := \frac{L}{\frac{S_n^{1-\beta}}{\mathrm{CES}(\mu_n^e)} + \frac{S_o^{1-\beta}}{\mathrm{CES}(\mu_o^e)}} \tag{36}$$

so that the dynamics of m_n and z are given jointly as (see proof in Appendix A.5):

$$\dot{m}_n = m_n(1-m_n)(\widehat{Q}_n - \widehat{Q}_o) + \alpha\beta z (m_n - \theta_n) \tag{37}$$

$$\dot{z} = z\Big(\alpha\beta z - m_n \widehat{Q}_n - (1-m_n)\widehat{Q}_o - \rho\Big) \tag{38}$$

Note that in (37) and (38), the aggregate quality growth rates $\widehat{Q}_j$ are given by (26), in which S_j are jointly pinned down by the static labour market condition.

3 Reduced Model

The dynamics of the benchmark model are fully determined by 3 macroeconomic equations for θ_n, z and m_n given by (32), (38) and (37) respectively, and 8 belief equations – $\tilde{M}_j$, $\tilde{\Sigma}_j$, v_j and B_j for both sectors j – as well as the static labour market clearing equation that solves for S_n at each point in time. These dynamics, as well as the algorithm for the calculation of higher moments of the beliefs distribution which appear for instance in (26), are described in Appendix A.6. Although the benchmark model is solvable through the simulation of these differential equations, we make the following set of assumptions in order to reduce the dimensions of the model, while retaining its key dynamic properties, i.e. multiplicity, path-dependency and the role of beliefs.

Assumption 2. From here on assume:

a. Learning externalities take the form $\eta(S_j) = \eta / S_j^{\gamma}$ where $\eta, \gamma \geq 0$ are constant parameters

b. The distribution of true productivities in each sector is a point mass: $\mu_{ji} = \overline{\mu}_j \ \forall i$.

c. The old sector is fully informed: $v_o = \Sigma_o = 0, \ M_o = \overline{\mu}_o \ \forall t$

Assumption 2.a imposes a functional form on the noise of the research process, and has the realistic implication that, as long as $\eta, \gamma > 0$, if all producers stop hiring scientists, i.e. $S_j \to 0$, then they receive infinitely noisy data and thus also stop learning about the productivity of their research process. In other words, a new sector can only learn about the productivity of its technology as long as it experiments with it. This functional form also includes the model without learning when $\eta = 0$. In this case firms' signal process has no noise and learning becomes infinitely fast, i.e. any initial distribution of beliefs converges immediately to the true productivity distribution. This formulation also includes the case where the speed of learning is fully independent of macroeconomic variables when $\gamma = 0$. In this case the amount of investment S_n in the new technology has no impact on the noise in firm' quality growth and thus the "learning block" of the model no longer depends on the macroeconomic block.

Assumption 2.b considers the limit case as the cross-sectional dispersion of firms' research productivity b^2 approaches zero. This is strictly a simplifying assumption in order to reduce the dimensions of the model and facilitate simulation. In particular, the first implication of this assumption is that now aggregate technology growth is given by

$$\widehat{Q}_j = \overline{\mu}_j S_j^{\beta} \frac{\mathbb{E}[(\mu_j^e)^{\frac{\beta}{1-\beta}}]}{\mathbb{E}[(\mu_j^e)^{\frac{1}{1-\beta}}]^{\beta}} \tag{39}$$

In equation (39) the correlation between beliefs μ_{ji}^e and true productivities μ_{ji}, which was previously described by the dynamic belief variable B_j, is no longer needed. Thus

Assumption 2.b also reduces the dimension of the belief dynamics to the three differential equations for precision of beliefs, v_j, and mean and variance of the cross-sectional belief distribution given by M_j and Σ_j respectively. These effects are summarised in lemma 2, with proof given in Appendix A.7.

Assumption 2.c further simplifies the model by removing the belief dynamics for the old sector, motivated by the idea that since that sector is more advanced, enough time has passed for the firms in that sector to fully learn their research productivity. Hence the focus is directed at beliefs about the new technologies.

Lemma 2. *Given Assumption (2), dynamic equilibrium is fully characterised by the following 6 differential equations and static labour market clearing condition:*

$$\dot{\theta}_n = \theta_n(1-\theta_n)(\sigma-1)(\widehat{Q}_n - \widehat{Q}_o) \tag{40}$$

$$\dot{m}_n = m_n(1-m_n)(\widehat{Q}_n - \widehat{Q}_o) + \alpha\beta z(m_n - \theta_n) \tag{41}$$

$$\dot{z} = z\Big(\alpha\beta z - m_n\widehat{Q}_n - (1-m_n)\widehat{Q}_o - \rho\Big) \tag{42}$$

$$\dot{v}_n = -\left(\frac{v_n S_n^{\gamma}}{\eta}\right)^2 \tag{43}$$

$$\dot{M}_n = \frac{v_n S_n^{2\gamma}}{\eta^2}(\overline{\mu}_n - M_n) \tag{44}$$

$$\dot{\Sigma}_n = \frac{v_n S_n^{2\gamma}}{\eta^2}(v_n - 2\Sigma_n) \tag{45}$$

$$1 = \frac{z}{m_n}\left(\frac{S_n}{\mathbb{E}[(\mu_n^e)^{\frac{1}{1-\beta}}]}\right)^{1-\beta} + S_n + \left(\frac{1-m_n}{m_n}\right)^{\frac{1}{1-\beta}} S_n \frac{\overline{\mu}_o^{\frac{1}{1-\beta}}}{\mathbb{E}[(\mu_n^e)^{\frac{1}{1-\beta}}]} \tag{46}$$

where $\widehat{Q}_j$ are given by

$$\widehat{Q}_n = \overline{\mu}_n S_n^{\beta} \frac{\mathbb{E}\left[(\mu_n^e)^{\frac{\beta}{1-\beta}}\right]}{\mathbb{E}\left[(\mu_n^e)^{\frac{1}{1-\beta}}\right]^{\beta}}, \quad \widehat{Q}_o = \overline{\mu}_o S_o^{\beta} = S_n^{\beta}\left(\frac{1-m_n}{m_n}\right)^{\frac{\beta}{1-\beta}} \frac{\overline{\mu}_o^{\frac{1}{1-\beta}}}{\mathbb{E}\left[(\mu_n^e)^{\frac{1}{1-\beta}}\right]^{\beta}} \tag{47}$$

3.1 Steady-States in the Reduced Case

The subsequent lemma outlines the steady-states of the model under Assumption 2, with proof outlined in Appendix A.8.

Lemma 3. *Suppose Assumption 2 holds. Then there are three steady-states:*

- *Two saddle-stable steady-states where only one sector is active. Let k and j be the active and inactive sectors, then in these steady-states $m_k = \theta_k = 1$, and $S_j = 0$ while S_k solves the implicit equation*

$$\alpha\beta\mu_k = \rho S_k^{1-\beta} + (1+\alpha\beta)\mu_k S_k \tag{48}$$

- *One unstable interior steady-state with simultaneous R&D in both sectors*

Note that in the new-only steady-state, $S_n > 0$, so learning takes place and in limit $M_n \to \overline{\mu}_n$ and $v_n, \Sigma_n \to 0$. But since $S_n = 0$ in the old-only steady-state, learning has stopped in the clean sector: $\dot{v}_n = \dot{\Sigma}_n = \dot{M}_n = 0$ is already satisfied, therefore all belief distributions (not necessarily point mass at $\overline{\mu}_n$) are consistent with the dirty-only steady-state

4 Application to the Energy Transition

Following the motivation presented in the introduction, in this section I calibrate the model to clean versus dirty energy-generation technologies and subsequently assess the effect of beliefs on the green transition. Unless otherwise stated, the parameters are given in Table 1.

α	ρ	σ	β	$\overline{\mu}_n$	$\overline{\mu}_o$	η	γ
1/3	0.005	3	3/4	0.09	0.09	0.09	0.5

Table 1: Parametrisation of the Reduced Case Model

In particular, the elasticity of substitution between clean and dirty technologies σ follows from the estimates of Papageorgiou, Saam and Schulte (2017). The curvature of the knowledge production function β is in line with the extensions in Bloom et al. (2020). The research productivities $\overline{\mu}_j$ are such that the long-run stable growth rate is 2.5% in both steady-states. The calibration of belief parameters η and γ is more difficult due to lack of empirical evidence on the speed of learning in such contexts. Equation (1) shows that the elasticity of learning with respect to aggregate investments is 2γ. Because of this $\gamma = 0.5$ is a reasonable calibration, implying that doubling the sector's research effort doubles the firms' speed of learning. Given this value of γ, I choose η such that the speed of learning in the model simulations conform with reasonable expectations.

It must be noted that η and γ not only describe belief dynamics in this model, but also the real process of firms' technology as in equation (4). This provides an additional dimension for calibration or robustness, namely on the volatility of firms' or sectors' technological growth. In this regard, the calibrations in table 1 conform to the general estimates of Farmer and Lafond (2016). They find that faster-growing sectors are more volatile. My calibrations are in line with this pattern: as shown by equation (25), individual firms' investment s_{ji} is linear in the sectoral research effort S_j. Therefore from equation (4), the net elasticity of technological volatility with respect to sectoral investments is $\beta - \gamma > 0$.

4.1 Dynamics with Perfect Information

We first consider the model without any belief distortions or heterogeneity, i.e. the mean beliefs of the new (clean energy) sector are also a point mass at the true productivity $\overline{\mu}_n$ and thus the dynamics of Proposition 2 reduce to 3-dimensions of θ_n, m_n and z. In this *no-belief* case the model corresponds closely to that of Smulders and Zhou (2022) as shown in the simulation of figure 4 except for simultaneous investment in both sectors. In particular, the projection of the dynamics on (θ_n, m_n)-space shows that for intermediate initial share of new/clean sector, the model is characterised by path multiplicity as there is an overlap between the saddle paths towards new-only and old-only steady-states. As aforementioned, the clean market share $\theta_{n,0}$ is historically conditioned, but since there is perfect foresight and the market valuation share m_n is forward-looking, each initial valuation share $m_{n,0}$, and thus each saddle path taken, is consistent with rational forward-looking expectations. Therefore in the interior of the overlap interval, $\theta_{n,0} \in (\underline{\theta}_n, \overline{\theta}_n)$, there are (at least) three self-fulfilling paths: one fast and one slow path to clean or dirty steady-states, and one path to the other, depending on the initial state of the economy.

In this model, as a consequence of diminishing marginal returns to research, there is simultaneous investment in both sectors up to the steady-state limit, as the trajectories for R&D effort S_j show. This means, for instance when the valuation share of the new technology is very small, there are still net positive returns to innovation in the new technology even though the economy may be on a path to full adoption of the old technology. The trajectories also showcase the extent of underinvestment in either technology due to externalities, especially in the transition period. Although the growth rate of the dominant stock of knowledge $\widehat{Q}_k$ approaches the balanced growth rate in all steady states, it is noticeably less during the transition. This is especially so for the slow saddle path where firms coordinate on the delayed adoption of the new technology.

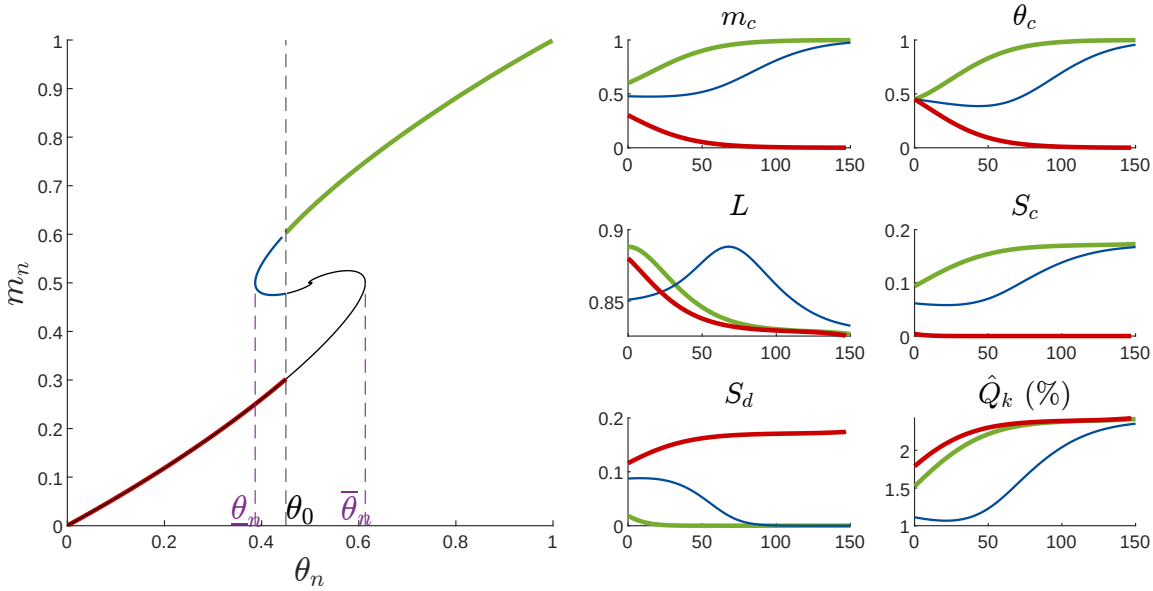


Figure 4: Fast (green) , slow (blue) and old/dirty (red) saddle paths the model without beliefs and trajectories from initial share of new/clean sector $\theta_{n,0} = 0.35$. The lower and upper boundaries of the overlap region are denoted by $\underline{\theta}_n$ and $\overline{\theta}_n$.

Furthermore as shown by Smulders and Zhou (2022) and in figure 5, the existence and size of the overlap region depends on the elasticity of substitution between the old and new technologies. In particular, innovation by firms within each sector entails two effects in the opposite direction: (1) the negative business-stealing effect as firms compete over market share of the sector and (2) the positive demand externality as a result of the knowledge externalities that shift spending towards the more innovative sector. Here if the two technologies are high enough substitutes (namely $\sigma > 2$) then the latter effect dominates the former and investments of firms within each sector become strategic complementarities leading to path multiplicity.

However, an important property of this model is that in the exterior of the overlap region the economy can only converge to a unique equilibrium. Specifically, the lower boundary $\underline{\theta}_n$ denotes the minimum initial share of the new or clean sector such that a transition to the new steady-state is possible, and the upper boundary θ_n denotes the minimum initial relative size such that a transition to the new steady-state is guaranteed. This has the implication that a new and relatively small sector cannot trigger a transition to a novel technology unless if that technology is much more productive compared to the old one, $\overline{\mu}_n >> \overline{\mu}_o$, as shown in figure 6.

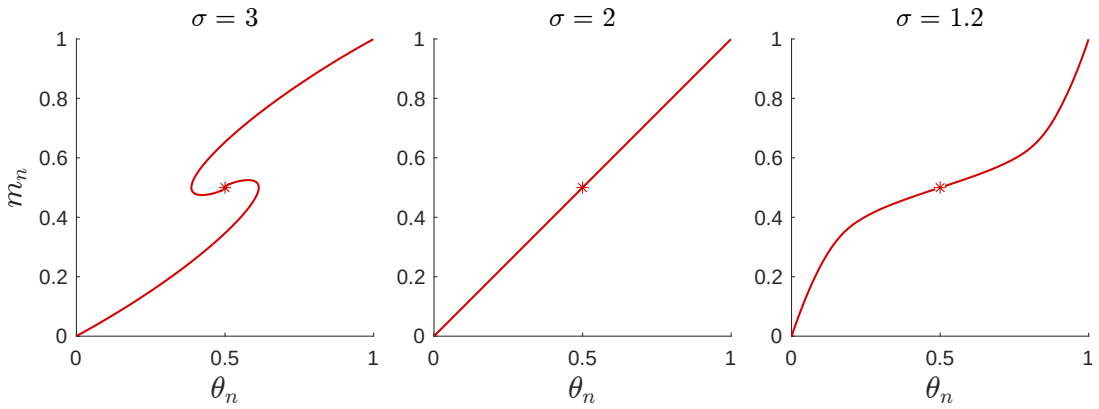


Figure 5: Saddle-path trajectories in (θ_c, m_c)-space, for the model without beliefs for different values of σ

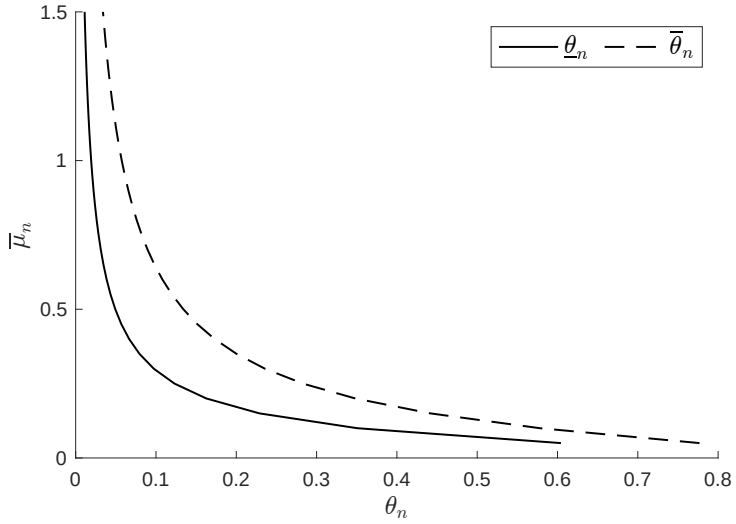


Figure 6: Boundaries of the overlap region for different values of $\overline{\mu}_n$

4.2 Dynamics with Beliefs and Learning

Given the previous observation that in the model without beliefs no transition to an initially small new technology is likely to take place, here we consider the alternative scenario with beliefs and learning. In particular we consider that the firms in the clean sector have an initial level of optimism $M_{n,0}$, confidence $1/v_{n,0}$ and disagreement $\Sigma_{n,0}$. Following proposition 1, this model is 6-dimensional but the effect of beliefs on the economy's trajectory and multiplicity can still be studied by focusing on the projection of the dynamics into (θ_n, m_n)-space.

It must be noted that in this model the initial distribution of prior beliefs is given exogeneously, even though at all times $t > 0$ the dynamics of beliefs are fully determined by endogenised Bayesian learning. Similar to the global games literature, this prior density can be considered as an initial idiosyncratic signal received by the firms,

which can be interpreted as the sector's noisy observation of the initial data from a new technology. That is, for instance, a new technology such as AI or Solar PVs has only started to be developed by a handful of firms, and given the uncertainty present in this early phase of technological development, as well as firms' protection of information, the sector at large receives a dispersed signal about its expected efficiency before it decides to invest or not.

Figure 7 shows the saddle paths and dynamics of the model with beliefs when starting from a moderately optimistic initial distribution of beliefs. The initially optimistic beliefs about clean technologies cause more clean-sector producers to innovate, thus increasing the initial amount of sustainable research labour S_n and increasing the speed of convergence to the clean steady-state when firms coordinate on the fast path. However, as firms learn about the true productivity over time, the amount of optimism declines and the economy converges to the case without beliefs.

More importantly, in contrast with the case without beliefs, as long as beliefs are optimistic enough, it is possible to transition to the clean steady-state even if the clean sector is initially small. As shown in Figure 8, when starting from an initially high amount of optimism about sustainable innovations, the economy can diverge from a close neighbourhood of dirty steady state, i.e. a low $\theta_{n,0}$, and undergo a full transition to the clean steady state. Due to the symmetries of the model, the converse implication is that with enough pessimism in clean technologies, the economy may diverge from a close neighbourhood of the new steady state towards the old-technology steady state. Therefore the ability of beliefs to extend the overlap region opens the possibility for drastic regime change. The mechanism in the emergence of new saddle paths due to (extreme) beliefs is that over time during these paths, investors learn and on average become more realistic about the prospects of the new technology, but in the meantime the new sector has grown large enough such that investing in it is ex post justified and a full transition is consistent with forward-looking rational behaviour.

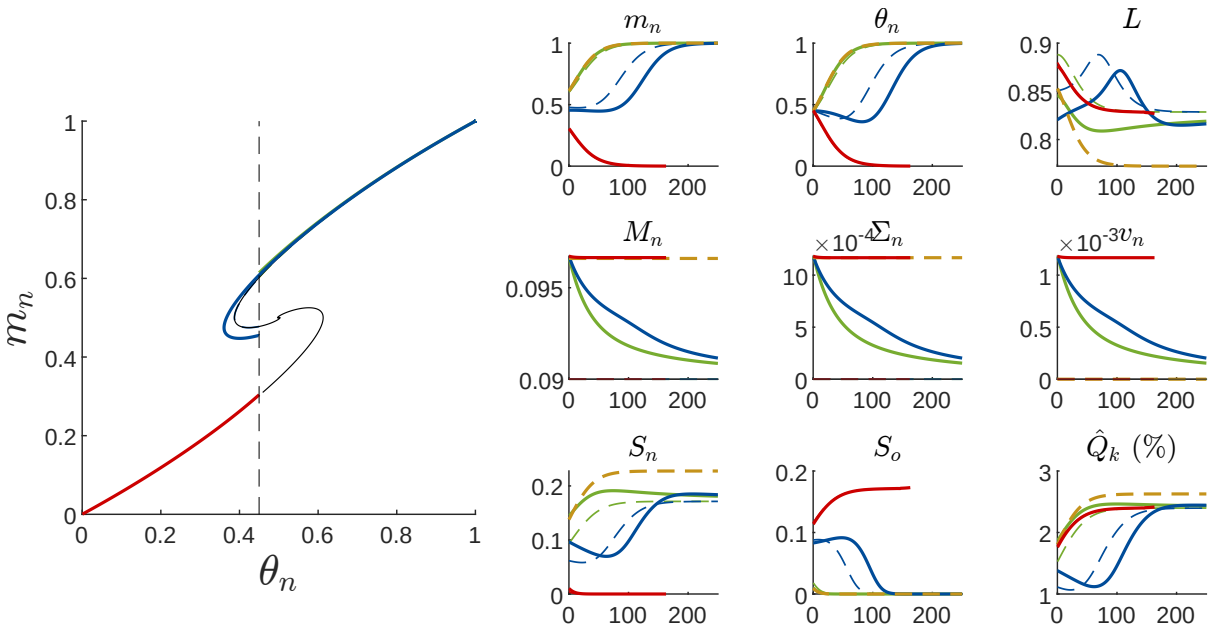


Figure 7: Fast (green), slow (blue), dirty (red) and no-learning (yellow) paths with moderate initial optimism. Dashed lines and solid black refer to the model without beliefs. *Parameters*: $\theta_0 = 0.45$, $M_{c,0} = 0.096$, $v_{c,0} = \Sigma_{c,0} = 0.0012$.

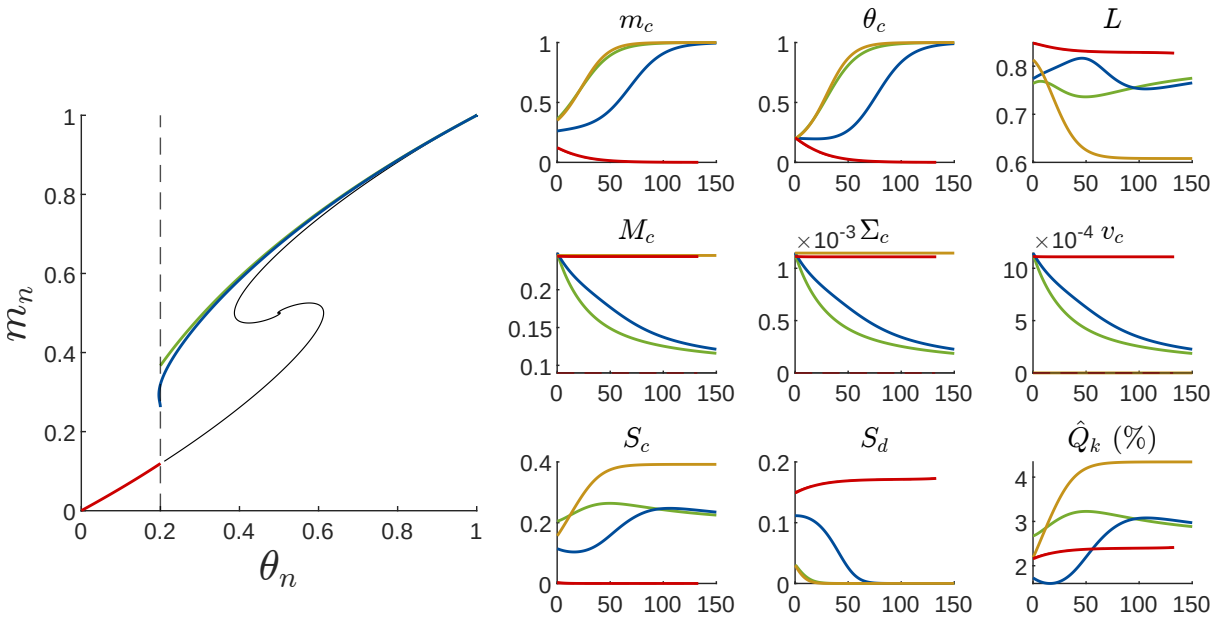


Figure 8: Fast (green), slow (dashed blue), dirty (red) and no-learning (yellow) paths with high initial optimism. Solid black refer to the model without beliefs. *Parameters*: $\theta_0 = 0.2$, $M_{c,0} = 0.25$, $v_{c,0} = \Sigma_{c,0} = 0.0012$.

Both figures 7 and 8 also show the trajectories for the *no-learning* counterfactual: where firms do not learn about their true productivity over time, perhaps due to initial full confidence of beliefs $v_{n,0} = 0$. Although an unlikely scenario, this counterfactual can be used to disentangle the role of learning in mitigating the effect of optimism and pessimism on the selection and growth of new technologies. In this case, because the

optimistic beliefs are permanent, the transition to the clean steady-state is faster and also there is more innovation, and thus higher TFP growth rate, in the steady-state itself. This is especially so for figure 8 with high optimism.

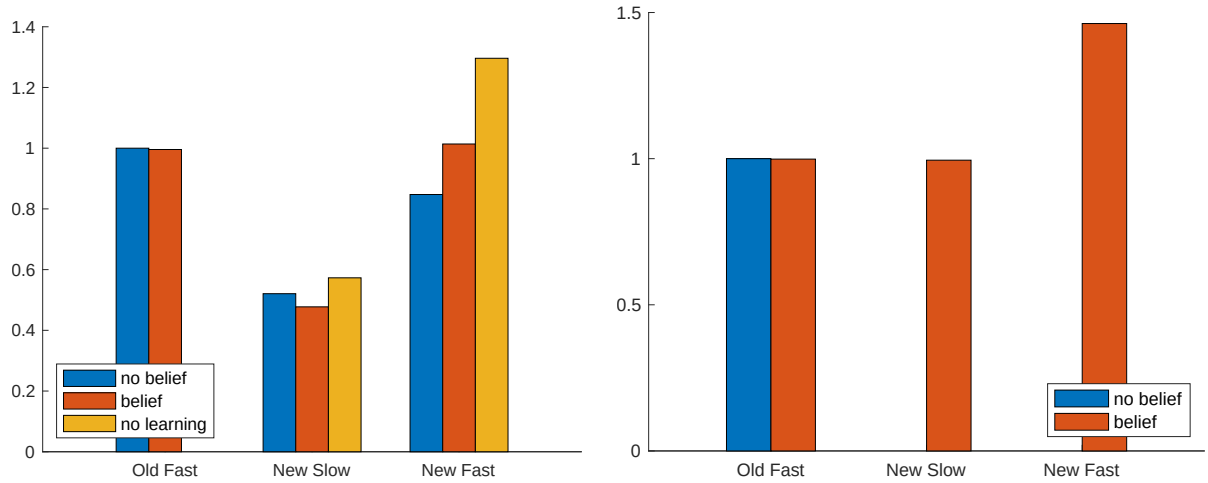
4.3 Welfare Gains from Optimism

Corresponding to simulations shown in Figures 7 and 8, Figure 9 shows the welfare effects of introducing optimism in the model, as compared to the old-only saddle path. Firstly we show that both moderate or high optimistic beliefs do not have a substantial impact on the welfare of the old-only path. This is because in both paths the amount of new technology investments is very small, therefore presence of beliefs about these technologies does not have a substantial impact on neither speed of convergence nor misallocation in these paths.

While no path to the new-only steady-state exists in the case with low initial share of new sector θ_n as shown in figure 8, it does exist in the case with high θ_n, albeit with lower welfare as compared to the old-only path. This is because in that scenario the economy is closer to the old steady-state and – in the absence of other externalities such as climate damages in this case – a faster convergence to the steady-state growth rate is welfare-improving. However, including optimism, almost surprisingly, entails higher welfare gains in the case with low initial θ_n. This is for two reasons. On one hand, a higher initial optimism results in a considerable increase in investments in the new technology which mitigates some of the endogeneous underinvestment in the model due to knowledge externalities – at least until the effect of learning dampens this effect. On the other hand, as shown in figure 3, the degree of misallocation is in fact lower in the presence of higher optimism. Taken together, these two effects imply that while moderate optimism does not have a significant effect on welfare when the new sector is already well-developed, high optimism can have significant welfare gains in the context of a small new technology.

These welfare gains of high optimism could be, however, lost if the economy coordinates on a slow path as shown in figure 9b. In this case, following the above reasoning and by definition of delayed investment, the recovery in investment as a result of optimism is only enough to compensate for the misallocation losses. In fact in the case with moderate beliefs there are welfare losses attributed to the coordination on a slow path. Finally, and not surprisingly, the counterfactual *no-learning* scenario results in welfare gains across the board[6]. This strengthens the result that the welfare gains from higher optimism, which bring investments closer to the socially optimal level, tend to dominate the welfare losses from misallocation as a result of heterogeneity in beliefs.

[6]Note that the no-learning welfare comparisons are omitted for the high optimism case as they are too large



(a) Moderate optimism

(b) High optimism

Figure 9: Welfare in permanent consumption equivalents. Reference scenario: fast path without beliefs to the old-only steady-state

4.4 Trade-offs in the Emergence of New Technologies

Figure 10 shows the boundaries of the overlap region for different initial levels of optimism $M_{n,0}$, while fixing initial confidence $v_{n,0}$, dispersion of beliefs $\Sigma_{n,0}$, as well as the true research productivity $\overline{\mu}_n$. Similar to figure 6, both boundaries are decreasing and convex in in the initial level of optimism. This means that given the initial share of new sector, enough optimism can make it possible, or even guarantee, a full transition towards the full adoption of the new technology. The mechanism is as before: while beliefs are still optimistic, the new sector grows large and thus even when beliefs have converged closer to the truth, the new technology is well-developed and a full transition is justified.

However learning causes a discontinuity: as the initial share of the new sector becomes smaller, an increasing amount of optimism is needed to make a full transition possible. Similarly, given moderate initial optimism, if the new sector is too small there may be no paths to the new-only steady state because investors learn fast enough and switch to old technologies. Another discontinuity in the model is at the extreme levels of pessimism: beyond a certain level of pessimism no path to the new steady-state exists even with very dominant initial new sector.

Figure 10 also shows the boundaries of the overlap region as a function of beliefs in the no-learning counterfactual case. Of particular interest in this case is the distance between the models with and without learning, namely the vertical difference in $M_{n,0}$ at each particular $\theta_{n,0}$. As the initial share of the new sector gets smaller, the increasing distance between this counterfactual and the model with learning shows the increasing burden of learning on the adoption of under-developed technologies. That is, with a smaller initial relative size of the new sector, a higher amount of optimism is required to facilitate or guarantee a transition, but the higher optimism also leads

to faster learning (at least in the beginning), which further necessitates higher initial optimism.

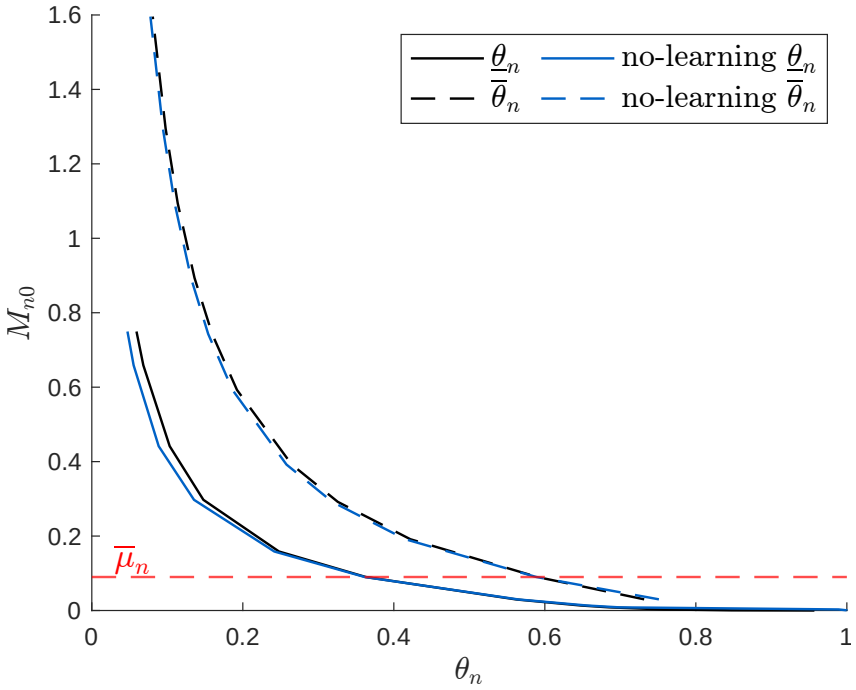


Figure 10: Overlap region boundaries as function of initial level of optimism, $M_{n,0}$, when $v_{n,0} = \Sigma_{n,0} = 0.03^2$

Similar to the model without beliefs, in addition to higher general optimism, a higher true productivity of new-technology innovation can also fascilitate the transition to the new steady-state. In other words, the transition towards the new technology can be made possible, or guaranteed, due to a combination of higher actual productivity of the new technologies or higher optimism about them. Related to this relationship, figure 11 shows, for a given small initial new sector, the minimum amount of initial optimism $M_{n,0}$ required for a transition to be possible (that is, when $\theta_{n,0} = \underline{\theta}_n$) for different values of actual productivity $\overline{\mu}_n$. As expected from the previous analysis, with enough optimism, a transition is possible even for low-productivity new technologies. In contrast with the model without beliefs, due to the small initial new sector a very high true productivity is required to make the transition possible. However, the convexity of the relationship shown in figure 11 implies that with each unit of decreasing productivity, an increasingly higher amount of optimism is required. This implies that in this model it is possible, but quite unlikely for a transition to an unproductive technology to take place.

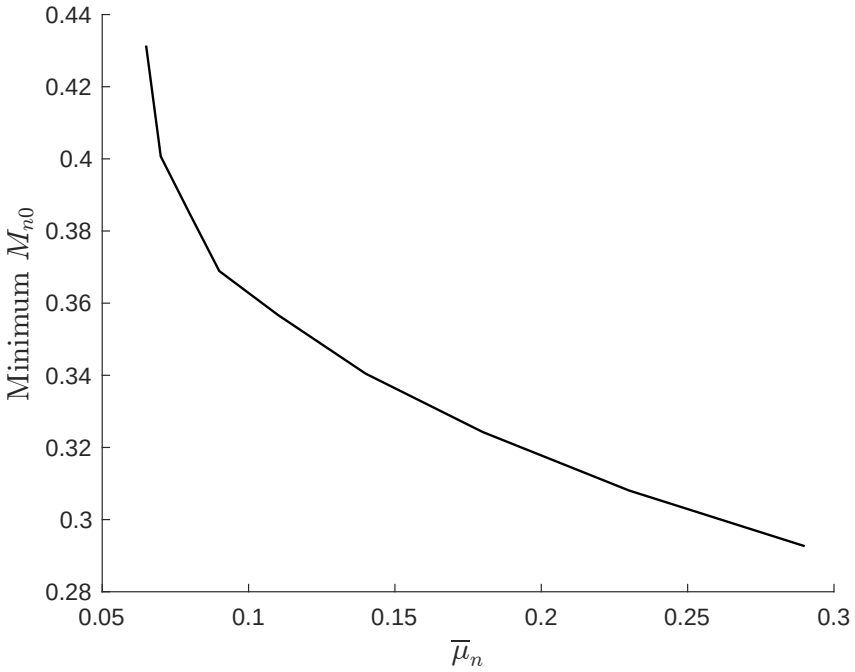


Figure 11: Minimum necessary beliefs $M_{c,0}$ such that a clean path exists, as function of actual productivity μ_c. $v_{c,0} = \Sigma_{c,0} = 0.023^2, \theta_{c,0} = 0.1$

5 Conclusion

I develop a directed technical change model with heterogeneous beliefs, learning, and strategic complementarities to explore how beliefs influence the adoption and diffusion of new technologies. I show that even in the presence of rational learning and the absence of higher-order beliefs, initial optimism or pessimism can shape the long-run technological trajectory. In particular, when beliefs about the productivity of new technologies are sufficiently optimistic, the economy can undergo a regime shift towards adopting these technologies, even when the new sector is initially small. This transition is driven by the interaction between beliefs and research investments, where optimism fosters early investment, allowing the new sector to grow while learning adjusts beliefs closer to the truth.

These results imply that beliefs can meaningfully affect the long-term technological landscape, particularly during critical transition periods. Expert projections and public sentiment – such as those from the International Energy Agency (IEA) in the context of the green transition – can significantly influence the speed and direction of technological change. Inaccurate pessimism may slow the adoption of beneficial technologies, while excessive optimism may direct investments toward less productive technologies, leading to inefficient outcomes. Therefore, understanding and managing beliefs about emerging technologies is crucial.

While some belief distributions can lead to uniqueness, in this model equilibrium multiplicity and path-dependency is otherwise retained by the fact that while each

firm is uninformed about its own productivity, the distribution of productivities and beliefs are common knowledge. This feature of the model enables a particular focus on the role of incomplete information and learning from experience, while maintaining tractability of distributions and consistency between private and common sources of information. However, without aggregate stochastics or incomplete information about aggregates and the future of the economy, there may be more than one equilibrium path consistent with rational forward-looking behaviour. Thus this model can act as a stepping-stone for future work that bridges this gap by including aggregate uncertainty or ambiguity in order to facilitate the selection of equilibria, possibly using a dynamic version of global games.

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A Appendices

A.1 Proof of Proposition 1: Distribution of Mean Belief

Ignoring sector j subscripts, the Kalman-Bucy filter equation is

$$\mathrm{d}\mu_i^e = \frac{v}{\eta(S)^2}(\mu_i - \mu_i^e)\mathrm{d}t + \frac{v}{\eta(S)}\mathrm{d}W_{it}$$

which using temporary notation can be written as a usual Ornstein-Uhlenbeck process

$$\mathrm{d}\mu_i^e = \theta_t(\mu_i - \mu_i^e)\mathrm{d}t + \sigma_t \mathrm{d}W_{it} \tag{49}$$

where now θ_t denotes the speed of mean-reversion of the OU process and σ_t the diffusion term. The difficulty in integrating the stochastic differential equation (49) is that the state variable μ_i^e appears on both sides of the equation. In order to solve this, we define an auxiliary function g as:

$$g(t) := \int_0^t \theta_s \mathrm{d}s$$

and function $f(\mu_i^e, t) := \mu_{it}^e e^{g(t)}$. Then by Ito's lemma the differential of f is given by the process

$$\begin{aligned}\mathrm{d}f &= \left[e^{g(t)}\theta_t(\mu_i - \mu_i^e) + \theta_t \mu_i^e e^{g(t)}\right]\mathrm{d}t + e^{g(t)}\sigma_t \mathrm{d}W_{it} \\ &= e^{g(t)}\theta_t \mu_i \mathrm{d}t + e^{g(t)}\sigma_t \mathrm{d}W_{it}\end{aligned}$$

Integrating this gives the solution

$$\begin{aligned} e^{g(t)}\mu_{it}^e - \mu_{i0}^e &= \int_0^t e^{g(s)}\mu_i \theta_s \mathrm{d}s + \int_0^t e^{g(s)}\sigma_s \mathrm{d}W_{is} \\ \iff \mu_{it}^e &= e^{-g(t)}\mu_{i0}^e + \mu_i e^{-g(t)}\int_0^t e^{g(s)}\theta_s \mathrm{d}s + e^{-g(t)}\int_0^t e^{g(s)}\sigma_s \mathrm{d}W_{is} \end{aligned} \tag{50}$$

which we can write as $\mu_{it}^e = A_t + B_t \mu_i$ where $B_t = e^{-g(t)}\int_0^t e^{g(s)}\theta_s \mathrm{d}s$ is deterministic and follows the law of motion

$$\dot{B}_t = \theta_t(1 - B_t), \quad B_0 = 0$$

which is equation (14) after replacing for θ_t.

And the stochastic variable $A_t = e^{-g(t)}\mu_{i0}^e + e^{-g(t)}\int_0^t e^{g(s)}\sigma_s \mathrm{d}W_{is}$ is independent of the distribution of true productivities μ_i, due to the assumption on independence of the initial distribution of beliefs from the true values, and since Brownian increments are by definition independent of both μ_{it}^e and μ_i. Since both μ_{i0}^e and Brownian increments are Gaussian, A_t also has a Gaussian distribution where

$$\begin{aligned} \tilde{M}_t &:= \mathbb{E}[A_t] = e^{-g(t)}M_0 \implies \dot{\tilde{M}}_t = -\theta_t \tilde{M}_t \\ \tilde{\Sigma}_t &:= \mathbb{V}\mathrm{ar}(A_t) = e^{-2g(t)}\tilde{\Sigma}_0 + e^{-2g(t)}\int_0^t e^{2g(s)}\sigma(s)^2 \mathrm{d}s \implies \dot{\tilde{\Sigma}}_t = -2\theta_t \tilde{\Sigma}_t + \sigma_t^2 \end{aligned}$$

which give equations (12) and (13) after replacing for θ_t and σ_t. Finally, the two equations $M_t = \tilde{M}_t + \overline{\mu} B_t$ and $\Sigma_t = \tilde{\Sigma}_t + b^2 B_t$ follow directly from taking expectation and variance of (50), thus completing the proof of Proposition 1.

A.2 Proof of Proposition 2: Linearity of Value Function

Firstly for ease of notation we ignore sector subscripts and use the shorthand

$$\pi(q_{it}, t) = \alpha(1-\alpha)\frac{P_t Y_t}{Q_t} q_{it} \equiv c_t q_{it}$$

to write the value function of firm i as

$$V(q_i, t; \mu_i^e, v) = \max_{\{s_u\}_{u=t}^{\infty}} \mathbb{E}\left\{ \int_t^{\infty} e^{-r(u-t)} [c_u q_{iu} - w_u s_u] \mathrm{d}u \,\middle|\, \mathcal{I}_{it}, q_{it} = q_i \right\} \tag{51}$$

$$\text{s.t. } \mathrm{d}q_{it} = s_{it}^{\beta} Q_t [\mu_i \mathrm{d}t + \eta_t \mathrm{d}W_{it}] \tag{52}$$

Where the information set $\mathcal{I}_{it}$ is defined in (8) and as explained in the main text, the firm takes μ_i^e and v as constants while in reality they follow (11) and (10).

Note that in the R&D decision problem (51)-(52), the expectation is taken both with respect to the unknown drift μ_i and the stochastic future value of q_i. Integrating (52) from time t to u gives

$$q_{iu} = q_{it} + \int_t^u s_{i\tau}^{\beta} Q_\tau \mu_i \mathrm{d}\tau + \int_t^u s_{i\tau}^{\beta} Q_\tau \eta_\tau \mathrm{d}W_{i\tau}$$

where we know the second element is a normally distributed random variable:

$$\int_t^u s_{i\tau}^{\beta} Q_\tau \eta_\tau \mathrm{d}W_{i\tau} \sim \mathcal{N}\left(0, \int_t^u \left(s_{i\tau}^{\beta} Q_\tau \eta_\tau\right)^2 \mathrm{d}\tau\right)$$

So the value function can be written as

$$\begin{aligned}
&V(q_i, t; \mu_i^e, v) = \\
&\max_{\{s_u\}_{u=t}^{\infty}} \mathbb{E}\left\{ \int_t^{\infty} e^{-r(u-t)} \left[c_u \left(q_{it} + \int_t^u s_\tau^{\beta} Q_\tau \mu_i \mathrm{d}\tau + \int_t^u s_\tau^{\beta} Q_\tau \eta_\tau \mathrm{d}W_{i\tau} \right) - w_u s_u \right] \mathrm{d}u \,\middle|\, \mathcal{I}_{it}, q_{it} = q_i \right\} \\
&= q_i \mathbb{E}\left[\int_t^{\infty} e^{-r(u-t)} c_u \mathrm{d}u \,\middle|\, \mathcal{I}_{it} \right] \qquad (53) \\
&+ \max_{\{s_u\}_{u=t}^{\infty}} \mathbb{E}\left\{ \int_t^{\infty} e^{-r(u-t)} \left[c_u \left(\int_t^u s_\tau^{\beta} Q_\tau \mu_i \mathrm{d}\tau + \int_t^u s_\tau^{\beta} Q_\tau \eta_\tau \mathrm{d}W_{i\tau} \right) - w_u s_u \right] \mathrm{d}u \,\middle|\, \mathcal{I}_{it} \right\} \qquad (54)
\end{aligned}$$

Furthermore, as seen from (10) and (11), the posterior belief about μ, summarised by μ_{it}^e and v_t, does not depend on q_i. So, the term (54) does not depend on q_i and from (53) it follows that V is linear in q_i, and the shadow value of changes in q is given by

$$\lambda_t := \frac{\partial V_{it}}{\partial q_i} = \mathbb{E}\left[\int_t^{\infty} e^{-r(u-t)} \alpha(1-\alpha) \frac{P_u Y_u}{Q_u} \mathrm{d}u \,\middle|\, \mathcal{I}_{it} \right]$$

Now since all aggregates and distributions are common knowledge across firms, all firms know the current and future values of P_j, Y_j and Q_j (and their individual decisions do not affect these aggregates since each firm is zero-measure). With this perfect foresight about aggregates we have

$$\lambda_t = \int_t^{\infty} e^{-r(u-t)} \alpha(1-\alpha) \frac{P_u Y_u}{Q_u} \mathrm{d}u$$

which immediately implies symmetry of $\partial V_{it}/\partial q_i$ across all firms and the differential equation (23).

The Bellman equation (22) relies on the above linearity result and perfect memory. The latter means that at each moment in time, an intermediate producer considers the entire past history of its quality process, i.e. $\mathcal{I}_{it_0} \subseteq \mathcal{I}_{it}\ \forall t_0 \leq t$. Other than these, the derivation of the Bellman equation is standard but is reproduced here for completeness. First, the value function at time t_0 can be written as

$$\begin{aligned} V(q_i, t_0; \mathcal{I}_i) &= \max_{\{s_u\}_{u=t_0}^{\infty}} \mathbb{E}\left\{ \int_{t_0}^{\infty} e^{-r(u-t_0)}[c_u q_{iu} - s_u w_u]\mathrm{d}u \,\middle|\, \mathcal{I}_{it_0} \right\} \\ &= \max_{\{s_u\}_{u=t_0}^{\infty}} \mathbb{E}\left\{ \int_{t_0}^{t} e^{-r(u-t_0)}[c_u q_{iu} - s_u w_u]\mathrm{d}u + \int_{t}^{\infty} e^{-r(u-t_0)}[c_u q_{iu} - s_u w_u]\mathrm{d}u \,\middle|\, \mathcal{I}_{it_0} \right\} \end{aligned} \tag{55}$$

Now

$$e^{-r(t-t_0)} V(q_i, t; \mathcal{I}_i) = \max_{\{s_u\}_{u=t}^{\infty}} \mathbb{E}\left\{ \int_{t}^{\infty} e^{-r(u-t_0)}[c_u q_{iu} - w_u s_u]\mathrm{d}u \,\middle|\, \mathcal{I}_{it} \right\}$$

and since $\mathcal{I}_{it_0} \subseteq \mathcal{I}_{it}$

$$\mathbb{E}\left[e^{-r(t-t_0)} V(q_i, t; \mathcal{I}_i) \,\middle|\, \mathcal{I}_{it_0} \right] = \max_{\{s_u\}_{u=t}^{\infty}} \mathbb{E}\left\{ \int_{t}^{\infty} e^{-r(u-t_0)}[c_u q_{iu} - w_u s_u]\mathrm{d}u \,\middle|\, \mathcal{I}_{it_0} \right\}$$

So (55) becomes

$$V(q_i, t_0; \mathcal{I}_i) = \max_{\{s_u\}_{u=t_0}^{\infty}} \mathbb{E}\left\{ \int_{t_0}^{t} e^{-r(u-t_0)}[c_u q_{iu} - s_u w_u]\mathrm{d}u \,\middle|\, \mathcal{I}_{it_0} \right\} + \mathbb{E}\left[e^{-r(t-t_0)} V(q_i, t; \mathcal{I}_i) \,\middle|\, \mathcal{I}_{it_0} \right]$$

Taking the differential of the above with respect to t gives

$$\begin{aligned} 0 &= \max_{s_t} \mathbb{E}\left[e^{-r(t-t_0)}(c_t q_{it} - s_t w_t) \,\middle|\, \mathcal{I}_{it_0} \right] + \frac{\mathbb{E}\left[\mathrm{d}\left(e^{-r(t-t_0)} V(q_i, t; \mathcal{I}_i) \right) \middle| \mathcal{I}_{it_0} \right]}{\mathrm{d}t} \\ &= \max_{s_t} \mathbb{E}\left[e^{-r(t-t_0)}(c_t q_{it} - s_t w_t) \,\middle|\, \mathcal{I}_{it_0} \right] + \frac{\mathbb{E}\left[e^{-r(t-t_0)}(-rV\mathrm{d}t + \mathrm{d}V) \middle| \mathcal{I}_{it_0} \right]}{\mathrm{d}t} \end{aligned} \tag{56}$$

where by Ito's lemma and the fact that V is linear in q

$$\begin{aligned} \mathrm{d}V &= \left[\frac{\partial V}{\partial q_i} s_i^{\beta} Q \mu_i + \frac{1}{2} \frac{\partial^2 V}{\partial q_i^2} s_i^{2\beta} Q^2 \eta^2 + \frac{\partial V}{\partial t} \right] \mathrm{d}t + \frac{\partial V}{\partial q_i} s_i^{\beta} \eta Q \mathrm{d}W_i \\ &= \left[\frac{\partial V}{\partial q_i} s_i^{\beta} Q \mu_i + \frac{\partial V}{\partial t} \right] \mathrm{d}t + \frac{\partial V}{\partial q_i} s_i^{\beta} \eta Q \mathrm{d}W_{it} \end{aligned} \tag{57}$$

Finally note that $\mathbb{E}[\mathrm{d}W_{it} \,|\, \mathcal{I}_{it_0}] = \mathbb{E}[\mathrm{d}W_{it}] = 0$ for $t_0 \leq t$. Using this when replacing (57) in (56), and taking the limit $t_0 \to t$ gives the Bellman equation, hence completing the proof of this proposition.

A.3 Proof of Lemma 1: Misallocation in R&D

For simplicity of notation define the random variables $X \equiv (\mu_{ji}^e)^{\frac{1}{1-\beta}}$ and $Y \equiv \mu_{ji}$ and function $\varphi(X,Y) = X^\beta Y$. Since φ is concave, by the vector form of Jensen's inequality we have

$$\mathbb{E}\left[(\mu_{ji}^e)^{\frac{\beta}{1-\beta}}\mu_{ji}\right] \equiv \mathbb{E}[\varphi(X,Y)] \leq \varphi(\mathbb{E}[X],\mathbb{E}[Y]) \equiv \mathbb{E}[(\mu_{ji}^e)^{\frac{1}{1-\beta}}]^\beta \mathbb{E}[\mu_{ji}]$$

and again using Jensen's inequality, we have

$$\mathbb{E}[\mu_{ji}] \leq \mathbb{E}[\mu_{ji}^{\frac{1}{1-\beta}}]^{1-\beta}$$

Thus

$$\widehat{Q}_j = S_j^\beta \frac{\mathbb{E}\left[(\mu_{ji}^e)^{\frac{\beta}{1-\beta}}\mu_{ji}\right]}{\mathbb{E}\left[(\mu_{ji}^e)^{\frac{1}{1-\beta}}\right]^\beta} \leq S_j^\beta \mathbb{E}[\mu_{ji}] \equiv \widehat{Q}_j^* \leq S_j^\beta \mathbb{E}[\mu_{ji}^{\frac{1}{1-\beta}}]^{1-\beta} \equiv \widehat{Q}_j^{**}$$

as required.

A.4 Derivation of Dynamic Allocation

To derive equation (35), first replace s_{ji} from (25) in the FOC for investments (24) to get

$$wS_j^{1-\beta} = \beta Q_j \lambda_j \text{CES}_{\frac{1}{1-\beta}}(\mu_j^e) \tag{58}$$

Since (58) is the same for both sectors, we have the following arbitrage equation:

$$w = \frac{\beta\lambda_n Q_n \text{CES}_{\frac{1}{1-\beta}}(\mu_n^e)}{S_n^{1-\beta}} = \frac{\beta\lambda_o Q_o \text{CES}_{\frac{1}{1-\beta}}(\mu_o^e)}{S_o^{1-\beta}}$$

Rearranging, using $m_n/m_o = \lambda_n Q_n/\lambda_o Q_o$ to get equation (35) as claimed.

A.5 Derivation of Dynamics of m_j and z

Firstly, rearranging (35) as

$$\frac{S_o^{1-\beta}}{\text{CES}(\mu_o^e)} = \frac{1-m_n}{m_n}\frac{S_n^{1-\beta}}{\text{CES}(\mu_n^e)}$$

and combining this with the definition (36), z can be written as

$$z = \frac{m_j L}{S_j^{1-\beta}/\text{CES}(\mu_j^e)}, \quad j \in \{n,o\} \tag{59}$$

Thus, replacing wages from (15) in (58) and using (33) we get

$$\frac{(1-\alpha)PY}{\beta\lambda_j Q_j} = \frac{L}{S_j^{1-\beta}/\text{CES}(\mu_j^e)} \equiv \frac{z}{m_j} \tag{60}$$

and from (23) in Proposition 2 and using (60) and (33) we get

$$\widehat{\lambda}_j = r - \alpha\beta\frac{\theta_j z}{m_j} \tag{61}$$

Dynamics of m_n: First, from equation (34) the law of motion of m_n is given by

$$\dot{m}_n = m_n(1-m_n)(\widehat{Q}_n - \widehat{Q}_o + \widehat{\lambda}_n - \widehat{\lambda}_o) \tag{62}$$

Here, the term $\widehat{\lambda}_n - \widehat{\lambda}_o$ can be simplified as follows using (61) and the fact that $\theta_n + \theta_o = 1$ and $m_n + m_o = 1$

$$\widehat{\lambda}_o = r - \alpha\beta\frac{(1-\theta_n)z}{1-m_n} \implies \widehat{\lambda}_n - \widehat{\lambda}_o = \alpha\beta z\left[\frac{1-\theta_n}{1-m_n} - \frac{\theta_n}{m_n}\right]$$

and replacing this in equation (62) gives the law of motion of m_n as claimed in (37).

Dynamics of z: The dynamics of z follow from capital market equilibrium. First, equation (60) in growth rates, gives

$$\widehat{z} = \widehat{m}_n + \widehat{P} + \widehat{Y} - \widehat{\lambda}_n - \widehat{Q}_n \tag{63}$$

Now combining (63) and (61) and (29) in capital market equilibrium, as well as the law of motion of m_n gives the law of motion of z in (38) as required.

A.6 Dynamics of Benchmark Model

The complete set of equations of the model are

$$\dot{\theta}_n = \theta_n(1-\theta_n)(\sigma-1)(\widehat{Q}_n - \widehat{Q}_o), \quad \theta_n(0) = \theta_{n,0} \tag{64}$$

$$\dot{m}_n = m_n(1-m_n)(\widehat{Q}_n - \widehat{Q}_o) + \alpha\beta z(m_n - \theta_n), \quad m_n(0) = m_{n,0} \tag{65}$$

$$\dot{z} = z\Big(\alpha\beta z - m_n\widehat{Q}_n - (1-m_n)\widehat{Q}_o - \rho\Big), \quad z(0) = z_0 \tag{66}$$

$$\dot{v}_j = -\frac{v_j^2}{\eta(S_j)^2}, \quad v_j(0) = v_{j,0}, \quad j \in \{n,o\} \tag{67}$$

$$\dot{\tilde{M}}_j = -\frac{v_j}{\eta(S_j)^2}\tilde{M}_j, \quad \tilde{M}_j(0) = M_{j,0}, \quad j \in \{n,o\} \tag{68}$$

$$\dot{\tilde{\Sigma}}_j = \frac{v_j}{\eta(S_j)^2}(v_j - 2\tilde{\Sigma}_j), \quad \tilde{\Sigma}_j(0) = \Sigma_{j,0}, \quad j \in \{n,o\} \tag{69}$$

$$\dot{B}_j = \frac{v_j}{\eta(S_j)^2}(1-B_j), \quad B_j(0) = 0, \quad j \in \{n,o\} \tag{70}$$

$$1 = \frac{z}{m_n}\left(\frac{S_n}{\mathbb{E}[(\mu_n^e)^{\frac{1}{1-\beta}}]}\right)^{1-\beta} + S_n + \left(\frac{1-m_n}{m_n}\right)^{\frac{1}{1-\beta}} S_n \frac{\mathbb{E}[(\mu_o^e)^{\frac{1}{1-\beta}}]}{\mathbb{E}[(\mu_n^e)^{\frac{1}{1-\beta}}]} \tag{71}$$

where $\widehat{Q}_j$ are given by (26):

$$\widehat{Q}_j = S_j^{\beta} \frac{\mathbb{E}\left[(\mu_{ji}^e)^{\frac{\beta}{1-\beta}} \mu_{ji}\right]}{\mathbb{E}\left[(\mu_{ji}^e)^{\frac{1}{1-\beta}}\right]^{\beta}} \tag{72}$$

and note that (71) is the static labour market clearing condition ($1 = L + S_n + S_o$) rewritten using the definition of z in (36) and the relationship between S_n and S_o in equation (35). Given m_n, z and the belief distribution, this equation solves for S_n at each point in time.

Calculation of Higher Moments

Note that in the labour market clearing condition (71) and growth rates (72), the higher moments of the belief distributions need to be calculated. However, since these distributions are normal, we can use existing equations (cf. Winkelbauer (2012)) for the higher moments $\mathbb{E}[(\mu_{ji}^e)^p]$ and $\mathbb{E}[(\mu_{ji}^e)^p \mu_{ji}]$ where p stands for $\frac{1}{1-\beta}$ or $\frac{\beta}{1-\beta}$. However for simplicity here we only consider the special case of integer p-moments, i.e. if $\beta \in \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$, in which case the moments have simpler expressions that will be described here.

Firstly, considering the beliefs about research productivity in the cross section of firms as a random variable, following Proposition 1 and Appendix A.1 it can be written as $\mu_{jit}^e = A_{jt} + B_{jt}\mu_{ji}$ where B_j is deterministic and follows (70) and $A_{jt} \sim \mathcal{N}(\tilde{M}_{jt}, \tilde{\Sigma}_{jt})$ independent of the true productivity μ_{jit}. Then, assuming p is an integer, the binomial expansion of $(\mu_{jit}^e)^p$ is

$$(\mu_{jit}^e)^p = \sum_{k=0}^{p} \binom{p}{k} A_{jt}^{p-k} B_{jt}^k \mu_{ji}^k$$

and so $(\mu_{jit}^e)^p \mu_{ji}$, which appears in $\widehat{Q}_j$ is

$$(\mu_{jit}^e)^p \mu_{ji} = \sum_{k=0}^{p} \binom{p}{k} A_{jt}^{p-k} B_{jt}^k \mu_{ji}^{k+1}$$

Thus, using the fact that A_{jt} and μ_{jit} are independent,

$$\mathbb{E}\left[(\mu_{jit}^e)^p\right] = \sum_{k=0}^{p} \binom{p}{k} \mathbb{E}\left[A_{jt}^{p-k}\right] B_{jt}^k \mathbb{E}\left[\mu_{ji}^k\right] \tag{73}$$

$$\mathbb{E}\left[(\mu_{jit}^e)^p \mu_{ji}\right] = \sum_{k=0}^{p} \binom{p}{k} \mathbb{E}\left[A_{jt}^{p-k}\right] B_{jt}^k \mathbb{E}\left[\mu_{ji}^{k+1}\right] \tag{74}$$

Now we can repeat the binomial expansion on A_{jt}^{p-k} and μ_{ji}^{ζ}, $\zeta \in \{k, k+1\}$. Namely, A_{ji} and μ_{ji} can be written as $A_{jt} = \tilde{M}_{jt} + \sqrt{\tilde{\Sigma}_{jt}} z_1$ and $\mu_{ji} = \overline{\mu}_j + b_j z_2$ with z_1, z_2 independent standard normal variables. For any Z standard normal and $n > -1$ we

know

$$\mathbb{E}[Z^n] = \begin{cases} 0 & \text{if} \quad n \text{ is odd} \\ \frac{n!}{(n/2)! \cdot 2^{n/2}} & \text{if} \quad n \text{ is even} \end{cases}$$

Therefore

$$\begin{aligned} \mathbb{E}\left[A_{jt}^{p-k}\right] &= \mathbb{E}\left[\left(\tilde{M}_{jt} + \sqrt{\tilde{\Sigma}_{jt}} z_1\right)^{p-k}\right] = \sum_{n=0}^{p-k} \binom{p-k}{n} \tilde{M}_{jt}^{p-k-n} \tilde{\Sigma}_{jt}^{n/2} \mathbb{E}[z_1^n] \\ &= (p-k)! \sum_{n=0}^{p-k} \frac{\mathbb{1}_{\{n \text{ even}\}} \tilde{M}_{jt}^{p-k-n} \tilde{\Sigma}_j^{n/2}}{(p-k-n)!(n/2)! \cdot 2^{n/2}} \end{aligned} \tag{75}$$

and for $\zeta \in \{k, k+1\}$

$$\mathbb{E}\left[\mu_{ji}^{\zeta}\right] = \mathbb{E}\left[\left(\overline{\mu}_j + b_j z_2\right)^{\zeta}\right] = \zeta! \sum_{n=0}^{\zeta} \frac{\mathbb{1}_{\{n \text{ even}\}} \overline{\mu}_j^{\zeta-n} b_j^n}{(\zeta-n)!(n/2)! \cdot 2^{n/2}} \tag{76}$$

Equations (73)-(76) fully characterise the higher belief moments that appear in the model dynamics.

A.7 Proof of Lemma 2: Dynamics of Reduced Model

Firstly, the belief dynamics (43), (44) and (45) follow directly from Proposition 1 and Assumption 2. Furthermore, the dynamics of θ_n, m_n, z and the labour market clearing condition follow directly from Appendix A.6. Finally equation (47) follows from imposing point mass true productivities

$$\mu_{ji} = \overline{\mu}_j, \quad \forall i, j$$

and full information in the dirty sector

$$\Sigma_{o,0} = v_{o,0} = 0, \quad M_{o,0} = \overline{\mu}_o$$

and using that when the mean belief density is a point mass at $\overline{\mu}_o$, we have of course

$$\mathbb{E}[(\mu_o^e)^p] = \int x^p \delta(x - \overline{\mu}_o \mathrm{d}x = \overline{\mu}_o^p$$

Calculation of Higher Moments

The calculation of higher moments in (46) and (47) are simplified compared to the benchmark model (described in section A.6). In particular in the case of integer p-moments, i.e. if $\beta \in \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$, we have

$$\begin{aligned} \mathbb{E}[(\mu_{ji}^e)^p] &= \mathbb{E}\left[\left(M_j + \sqrt{\Sigma_j} Z\right)^p\right] = \sum_{k=0}^{p} \binom{p}{k} M_j^{p-k} \Sigma_j^{k/2} \mathbb{E}[Z^k] \\ &= p! \sum_{k=0}^{p} \frac{\mathbb{1}_{\{k \text{ even}\}} M_j^{p-k} \Sigma_j^{k/2}}{(p-k)!(k/2)! \cdot 2^{k/2}} \end{aligned} \tag{77}$$

Equation (77) fully characterises the higher belief moments that appear in the reduced model dynamics.

A.8 Proof of Lemma 3: Steady States in the Reduced Case

The three steady-states of the model arise from first setting $\dot{\theta}_n = 0$ in equation (40).

Simultaneous Steady-State In this case, $\theta_n \notin \{0, 1\}$, so from $\dot{\theta}_n = 0 \implies \widehat{Q}_n = \widehat{Q}_o$. Therefore, in this case there is investment in both clean and dirty sectors (we ignore the uninteresting case of no investment in any sector) and using (35) we can determine m_n in this steady-state

$$\begin{aligned} \overline{\mu}_n S_n^\beta = \overline{\mu}_o S_o^\beta = \overline{\mu}_o S_n^\beta \left(\frac{1-m_n}{m_n}\right)^{\frac{\beta}{1-\beta}} \frac{\overline{\mu}_o^{\frac{\beta}{1-\beta}}}{\mathbb{E}\left[\mu_n^{\frac{1}{1-\beta}}\right]^\beta} \\ \iff m_n = \left[\left(\frac{\overline{\mu}_n}{\overline{\mu}_o}\right)^{1/\beta} \frac{\text{CES}_{\frac{\beta}{1-\beta}}(\mu_n^e)}{\overline{\mu}_n} + 1\right]^{-1} \end{aligned} \tag{78}$$

However note that since in this steady-state there is asymptotic investment in the clean technology (i.e. $S_n > 0 \; \forall t$), and since agents are not fully confident at any point $v_{n,0} > 0$, learning always takes place and the beliefs of the clean sector converge uniquely to the true new technology research productivity: $M_n \to \overline{\mu}_n$, $v_n \to 0$ and $\Sigma_n \to 0$. Therefore (78) in this steady-state simplifies to

$$m_n = \left[\left(\frac{\overline{\mu}_n}{\overline{\mu}_o}\right)^{1/\beta} + 1\right]^{-1} \tag{79}$$

However, equation (78) can still be used in the counterfactuals without learning (i.e. setting $v_{n,0} = 0$). Furthermore, $\dot{m}_n = 0$ implies $\theta_n = m_n$ and since beliefs converge to the true productivity, from equation (47) the growth rate in both sectors is given by $\widehat{Q}_j = \overline{\mu}_j S_j^\beta$. Setting $\dot{z} = 0$ implies, using $\widehat{Q}_n = \widehat{Q}_o$, that z in this steady-state is given by

$$z = \frac{\rho + \widehat{Q}_n}{\alpha\beta} = \frac{\rho + \overline{\mu}_n S_n^\beta}{\alpha\beta}$$

Using this, the labour market clearing condition (46) becomes

$$\begin{aligned} 1 &= \frac{\rho + \overline{\mu}_n S_n^\beta}{\alpha\beta m_n \overline{\mu}_n} S_n^{1-\beta} + S_n + \left(\frac{1-m_n}{m_n}\right)^{\frac{1}{1-\beta}} S_n \left(\frac{\overline{\mu}_o}{\overline{\mu}_n}\right)^{\frac{1}{1-\beta}} \\ &= \frac{\rho + \overline{\mu}_n S_n^\beta}{\alpha\beta\overline{\mu}_n} S_n^{1-\beta} \left[\left(\frac{\overline{\mu}_n}{\overline{\mu}_o}\right)^{1/\beta} + 1\right] + S_n + S_n \left(\frac{\overline{\mu}_n}{\overline{\mu}_o}\right)^{1/\beta} \end{aligned} \tag{80}$$

The nonlinear equation (80) can be shown to have a unique solution for S_n in the simultaneous steady-state and thus also determine z.

New-Only Steady-State In this case $\theta_n = 1$, then from $\dot{m}_n = 0$ we have $m_n = 1$. Therefore in this steady state all technology investment is in the new sector $S_o = 0, S_c > 0$ and therefore, like the previous case, the new-sector beliefs converge to the truth. Also similar to the previous case, from $\dot{z} = 0$ we have

$$z = \frac{\rho + \widehat{Q}_n}{\alpha\beta} = \frac{\rho + \overline{\mu}_n S_n^{\beta}}{\alpha\beta}$$

and the labour market clearing simplifies to

$$\alpha\beta\overline{\mu}_n = \rho S_n^{1-\beta} + (1 + \alpha\beta)\overline{\mu}_n S_n$$

which, again, has a unique solution for S_n.

Old-Only Steady-State In this case $\theta_n = 0$, then from $\dot{m}_n = 0$ we have $m_n = 0$. Therefore in this steady state all technology investment is in the old sector $S_n = 0, S_o > 0$ and therefore, unlike the previous cases, the new-sector beliefs do not converge to the truth. This means that this steady-state is consistent with any distribution of beliefs about the new sector.

However the non-convergence of new-sector beliefs does not affect the macroeconomic variables of this steady-state since the size of the new sector is 0 here. Therefore, like the new-only steady-state, from $\dot{z} = 0$ we have

$$z = \frac{\rho + \widehat{Q}_o}{\alpha\beta} = \frac{\rho + \overline{\mu}_o S_o^{\beta}}{\alpha\beta}$$

and the labour market clearing simplifies to

$$\alpha\beta\overline{\mu}_o = \rho S_o^{1-\beta} + (1 + \alpha\beta)\overline{\mu}_o S_o$$

which, again, has a unique solution for S_o.